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RESEARCH ARTICLE

No-arbitrage, state prices and trade in thin financial markets

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Abstract We examine how non-competitiveness in financial markets affects the choice of asset portfolios and the determination of equilibrium prices. In our model, potential arbitrage is conducted by a few highly specialized institutional investors who recognize and estimate the impact of their trades on financial prices. We apply a model of economic equilibrium, based on Weretka (http://www.ssc.wisc.edu/~mweretka/Research, 2007a), in which price effects are determined endogenously as part of the equilibrium concept. For the case in which markets allow for perfect insurance, we argue that the principle of no-arbitrage asset pricing is consistent with non-competitive behavior of the arbitragers and extend the fundamental theorem of asset pricing to the non-competitive setting.

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0 Introduction

The model of competitive equilibrium imposes the premise that all agents intervening in the markets observe a price that they take as given: they assume that their trade in the market, however large, can be made at that fixed price. In finance, the theory of the determination of the prices of financial derivatives, and the formulæ derived from this theory and applied in practice, are based on the premise that asset prices should permit no costless and riskless opportunities for profit; under this condition, asset prices embed an "objective" probability distribution that can be used to obtain the price of any asset as its discounted expected return: this is the basic content of the Fundamental Theorem of Asset Pricing, of its applications (the Modigliani–Miller theorem, for instance), and of the formulæ developed to price financial derivatives (for example, the Black-Scholes formula). When the competitive model is applied in a setting of uncertainty and trade of financial assets, the premise of finance holds at equilibrium: under minimal assumptions, at competitive equilibrium prices there should be no arbitrage opportunities, and hence the theory of asset pricing is immediately applicable.

The assumption that all traders ignore the impact of their trades on prices, however, appears at odds with empirical evidence from crucial markets. In financial markets, even as deep as NYSE or NASDAQ, the evidence is clear in this direction: large institutional traders have non-negligible price impacts, which they do take into account when designing their trading strategies, and tools have been developed to implement portfolio recompositions taking into account their price impacts.[1]

Attempts to develop a general equilibrium framework in which traders do take into account their price impacts have so far proved to be less successful than their competitive counterpart. The work initiated by Negishi (1961), though interesting, either required the imposition of very strong assumptions, or gave highly complex, intractable and inapplicable models: in the Negishi's approach, traders were allowed to have subjective perceptions of the price impact, which implied a great deal of arbitrariness on the determination of equilibria, unless the subjective conjectures where exogenously restricted; later attempts to impose more structure to the definition of equilibrium endowed the model with explicit strategic games, which came at the cost of tractability.[2]

In this paper, we develop a model of complete thin financial markets. As in Shleifer and Vishny (1997), we take the view that in many markets only a relatively few highly specialized institutional investors have capacity to arbitrate prices, and that

[1] For literature on this, see Keim and Madhavan (1996), Chan and Lakonishok (1995) and Perold (1988). Importantly, these large traders constitute a very significant portion of volume of trade in financial markets: see Gompers and Metrick (2001).

[2] For a review of this literature, see Hart (1985).

trade among such investors is the key price determinant. Shleifer and Vishny (1997) consider a dynamic (three-period) model with one risky asset and price-taking arbitrageurs. They depart from the frictionless setting by assuming that arbitrageurs, who are perfectly informed about the fundamental value of an asset, can take positions against the mispricing generated by noise traders using only (uninformed) investors' capital. In the agency model, Sheifer and Vishny show that, in scenarios with prolonged asset mispricing, investors may lose confidence in arbitrageurs' ability and refuse to provide them with liquid capital. This can reduce the effectiveness of arbitrage. Our model differs from the one proposed by Shleifer and Vishny in a number of respects: first, we consider a model in which assets are traded only once; second, we endow our market with a complete set of assets in which arbitrageurs are not constrained by capital; also, unlike in Shleifer and Vishny, we study cross-asset rather than cross-period arbitrage. [3] We depart from the classic model by relaxing the price-taking assumption, a mechanism different from the one studied by Shleifer and Vishny, and in this sense this paper complements their analysis, by studing how the presence of price impact affects asset prices and arbitrage opportunities, even in the absence of information asymmetries.

Our model is also related to the work of Zigrand (2004, 2006) who, in a Cournot–Walras setting, studies arbitrage across segmented markets, each with multiple assets. In Zigrand (2004, 2006), in each of the individual markets, there are no arbitrage opportunities across assets no arbitrage, and the existence of pricing kernel straightforwardly follows from the presence of price taking traders, but there may be arbitrage opportunities for agents who can trade across markets. In this paper, we focus on one market with all large, and hence non-competitive, arbitrageurs. Our main result shows that no-arbitrage and pricing kernel are consistent with equilibrium, even without the presence of perfectly competitive traders.

We apply a model of economic equilibrium, based on Weretka (2007a), in which all individual traders recognize the impact that their trades have on financial prices, and in which these effects are determined endogenously as part of the equilibrium concept.[4] This model maintains the flexible, tractable and spare structure of the competitive model, but the departure from the competitive premise implies that none of the standard positive or normative results are immediately applicable, and, in particular, the necessity of a no-arbitrage condition, and the derivation of an objective probability have to be argued. We show that, even in the absence of competitive behavior, the principle of no-arbitrage is consistent with equilibrium pricing in financial markets, and we extend the Fundamental Theorem of Asset Pricing to the noncompetitive setting; to the best of our knowledge, there is no previous study of noncompetitive markets with many arbitrary assets that possibly have collinear payoffs.[5] In this paper, we concentrate on the case in which the existing financial markets allow for complete insurance against risk.

[3] The dynamic arbitrage with non-competitive traders is studied in detail in Rostek and Weretka (2009).

[4] This is the feature that closes the definition of equilibrium imposing interesting structure.

[5] In the important literature on financial microstructure, starting with Kyle (1989), price impacts of the traders result from asymmetric information, rather than from the thinness of the markets. We believe that our work is complementary to this literature.

Our conclusions differ from the ones obtained by Koutsougeras (2003) and Koutsougeras and Papadapoulos (2004), where they exhibit noncompetitive markets whose equilibrium prices allow for arbitrage opportunities. This result holds for Nash equilibria, under the assumption that the assets are traded on strategic market games, *a-la* Shapley and Shubik (1977). Our concept of equilibrium is different, in a sense that is explained below, and allows for the possibility that each trader believes that her trades have cross-market price effects, while Nash behavior in Shapley-Shubik games does not.

This paper is part of a research program in which we study the properties of abstract thin markets. As a part of this program, Weretka (2007b) shows that the concept of equilibrium from this paper refines a Subgame Perfect Nash equilibrium in the game with demand schedules (Walrasian Auction). Our framework thus generalizes strategic models based on symmetric Linear Bayesian Nash equilibirum[6] to environments with heterogenous, non-quadratic utility functions and many, possibly redundant assets. There, it is also argued that the equilibrium outcome will be observed in anonymous markets, in which investors have no other information but their past trades and market prices, and therefore they discover market power through a statistical inference by estimating slopes of their demands using Market Impact Models. Therefore the equilibrium has a game theoretic foundation endowed with the learning process by which anonymous financial markets converge to this equilibrium. We believe that the assumption of anonymity is a realistic assumption in financial markets, in which large institutional investors know that they have price impacts, but often have no information about those with whom they are trading. Finally, Weretka (2007a) demonstrates that price impacts of the traders become negligible in economies with large number of traders and hence our thin market model provides a foundation for a competitive complete market model.

1 The Economy

Consider a financial market that evolves over two periods. Suppose that in the second period there is a finite number of states of the world, indexed by $s = 1, \ldots, S$, while in the first period investors trade assets that deliver a return in the second period. There is only one good, the numèraire, and all assets are denominated in this commodity. There is a finite number of assets, indexed by $a = 1, \ldots, A$, and their payoffs are uncertain: in the second period, asset a pays R_s^a in state s. Vector $R^a = (R_1^a, \ldots, R_S^a)^\top$ is a commonly known random variable, and matrix $R = (R^1, \ldots, R^A)$, which is of dimensions $S \times A$, denotes the financial structure of the economy.[7] We restrict our attention to complete financial markets: we assume that there are at least as many assets with linearly independent payoffs as states of the world. Then, without further loss of generality, we assume that the first S assets are linearly independent: we partition $R = (R_S, R_{A-S})$, and assume that the $S \times S$ matrix R_S is nonsingular.

[6] See Kyle (1989) or Vayanos (1999). Our model is more restrictive than the one considered in these two papers, in that in our model there is no asymmetric information.

[7] If we denote by $\mathbf{I}_n$ the $n \times n$ identity matrix, then $R = \mathbf{I}_S$ is the case of a full set of elementary securities.

There is a finite number of investors, who trade assets among themselves; a typical investor is indexed by $i = 1, \ldots, I$. Investor i is endowed with a future wealth of $e_s^i > 0$, contingent on the state of the world, and we denote by $e^i = (e_1^i, \ldots, e_S^i)$ the random variable representing investor i's future endowment of wealth. Each trader has preferences with respect to her wealth (or consumption) in the second period, with cardinal utility indices $u_s^i : \mathbf{R}_{++} \rightarrow \mathbf{R}$. We assume that each utility index is twice continuously differentiable, differentiably strictly monotonic and differentiably strictly concave, and satisfies Inada conditions. For notational simplicity, henceforth we write $u^i((c_s^i)_{s=1}^S) = \sum_s u_s^i(c_s^i)$; by our assumptions, $\partial u^i \gg 0$,[8] while $\partial^2 u^i$ is a diagonal, negative definite matrix. Given the assumptions, our model accommodates standard von Neumann–Morgenstern preferences in which expected utility for each trader is determined for arbitrary probabilistic beliefs, that can be heterogenous across traders.

The economy is completely characterized by the triple of a profile of individual preferences, a profile of future endowments of wealth and an asset payoff structure, namely $\{u, e, R\} = \{(u^i)_{i=1}^I, (e^i)_{i=1}^I, R\}$.

2 Equilibrium in thin markets

As there is only one commodity, trade takes place only in the market for assets in the first period. Asset prices are denoted by $P = (P_1, \ldots, P_A)^\top$, and the portfolio of trader i is denoted by Θ^i.[9] For each investor i, let $M_{a,a'}^i$ measure individual her belief of how the price of asset a changes when she expands her buy order for asset a' by one share. In the competitive model, this measure is exogenously fixed at zero; here, we follow Weretka (2007a) in letting $M_{a,a'}^i$ be endogenously determined as part of the equilibrium concept. The $A \times A$ matrix

$$M^i = \begin{pmatrix} M_{1,1}^i & \cdots & M_{1,A}^i \\ \vdots & \ddots & \vdots \\ M_{A,1}^i & \cdots & M_{A,A}^i \end{pmatrix} \tag{1}$$

will be referred to as trader i's *price impact matrix*: she believes that the price change exerted by a perturbation Δ to her portfolio is $M^i \Delta$. We will restrict attention only to positive definite and symmetric price impact matrices, and it will be a part of the definition of equilibrium that the profile of individual matrices is mutually consistent, in the sense that each matrix M^i correctly measures the price impacts of trader i, given the individual price impacts perceived by all other traders in the market.

Intuitively, an equilibrium in thin financial markets will be a triple consisting of a vector of asset prices, a profile of portfolios and a profile of price impact matrices, $(\bar{P}, \bar{\Theta}, \bar{M})$, such that:

[8] We take this vector as a column.

[9] This means that the future numèraire wealth of investor i is given by $e^i + R\Theta^i$, while the cost of her portfolio, if she constitutes it at prices $\tilde{P}$, is $\tilde{P} \cdot \Theta^i$, which she incurs in the first period.

1. all markets clear;
2. all traded portfolios are individually optimal, given the price impacts perceived by agents: for each trader i, at prices $\bar{P}$, trade $\bar{\Theta}^i$ is optimal, given that any other trade, $\tilde{\Theta}^i$, would change prices to $\bar{P} + \bar{M}^i(\tilde{\Theta}^i - \bar{\Theta}^i)$; and
3. all perceived price impact matrices correctly estimate the effects of individual portfolio perturbations on the prices that would be required for the rest of the market to optimally absorb them: for each trader i, (small) perturbations to her portfolio, Δ^i, uniquely define a price at which the other traders of the market, given their price impacts, are willing to supply Δ^i to i, and the derivative of this mapping at equilibrium trade $\bar{\Theta}^i$ is $\bar{M}^i$.

The first condition in the definition is standard, as in the definition of competitive equilibrium; the other two are not, so we now proceed to their formal presentation.

2.1 Stable individual trade

Here, each trader perceives that her trade affects prices, and takes these impacts into account when designing her investment strategy. We assume that trader i wants to maximize the ex-ante utility of her future wealth, net of the present cost of constituting her portfolio.[10] Critically, when computing the cost of constituting her portfolio, trader i will take into account (her perception of) the effect that her portfolio has on prices. To avoid confusion with the competitive model, we refer to this condition as *stability*.

Definition 1 Given prices P and a price impact matrix M^i, portfolio $\bar{\Theta}^i$ is *stable* for individual i if it solves program

$$\max_{\Theta^i} -(P + M^i(\Theta^i - \bar{\Theta}^i)) \cdot \Theta^i + u^i(e^i + R\Theta^i). \tag{2}$$

The second condition in the definition of equilibrium will be the requirement that, given their beliefs on price impacts, no trader will find that she can increase her (net) ex-ante utility by demanding a different portfolio: trader i's portfolio should be stable given equilibrium prices and her perceived price impacts at equilibrium trade.[11] It

[10] Formally, we are assuming that individual preferences are quasilinear on the wealth of the first period. We will not impose non-negativity constraints on this variable, hence we do not need to specify individual endowments of present wealth.

[11] Suppose that trader i is faced with the inverse demand function

$$p(\Theta) = P + M^i(\Theta - \bar{\Theta}) \cdot \Theta \tag{3}$$

from the rest of the market, where $\bar{\Theta}$ is some reference point. For each $\bar{\Theta}$, let

$$\Gamma^i(\bar{\Theta}) = \text{argmax}_{\Theta} - (P + M^i(\Theta - \bar{\Theta})) \cdot \Theta + u^i(e^i + R\Theta). \tag{4}$$

The stability condition requires that $\bar{\Theta}^i$ be a fixed point of this mapping.

requires that for each trader, given her price impact, no deviation from the equilibrium trade be able improve her ex-ante utility.[12]

Given the assumption of strong concavity of preferences and the condition that price impact matrices are to be positive definite, stability of portfolio $\bar{\Theta}^i$, given (P, M^i), is characterized by the condition that

$$P + M^i \bar{\Theta}^i = R^\top \partial u^i (e^i + R\bar{\Theta}^i). \tag{5}$$

2.2 Subequilibrium and consistent price impact matrices

The third condition of the definition of equilibrium requires the perceived price impacts of all traders to be accurate estimates of their actual price impacts, at least locally and to a first order, given the perceived price impacts of every other trader. Unlike in a competitive market, each trader here knows that prices must endogenously adjust to encourage other traders to (optimally) accommodate her trade. Consequently, in this model markets clear and other investors respond optimally to prices, *even* if one of the investors is trading a suboptimal portfolio; this feature is captured by the notion of subequilibrium in financial markets, defined below.

Definition 2 Given an individual i and a profile of price impact matrices for all traders but i, $M^{-i} = (M^j)_{j \neq i}$, a *subequilibrium triggered by trade* $\tilde{\Theta}^i$ is a pair of a vector prices and a profile of trades by all other investors, $(\tilde{P}, \tilde{\Theta}^{-i})$, such that:

1. all markets clear: $\tilde{\Theta}^i + \sum_{j \neq i} \tilde{\Theta}^j = 0$;
2. all traders other than i act optimally: for each $j \neq i$, trade $\tilde{\Theta}^j$ is stable given $(\tilde{P}, M^j)$.

For each i, let $\mathcal{S}^i(\tilde{\Theta}^i; M^{-i})$ be the set of all subequilibria triggered by trade $\tilde{\Theta}^i$, given that the other traders in the market perceive their price impacts as M^{-i}, and let $\mathcal{P}^i(\tilde{\Theta}^i; M^{-i})$ be the projection of that set into the space of prices. If $\mathcal{P}^i(\tilde{\Theta}^i; M^{-i})$ is singleton-valued in a neighborhood of some $\bar{\Theta}^i$, it defines, at least locally, an inverse demand function, $P^i(\tilde{\Theta}^i)$, faced by trader i given M^{-i}. If such function is differentiable, trader i can estimate its derivatives. The condition of consistency of price impact matrices is that, at equilibrium, perceived price impacts should not be arbitrary: each trader is able to estimate the slope of her real inverse demand, so that the inverse demand function implicit in the condition of stability of her trade is a correct first-order approximation to her real inverse demand, given the estimations made by all other traders.[13] This concept of mutual consistency of price impact estimations, which completes the definition of equilibrium, is formalized next.

[12] Of course, the trader can always feasibly trade nothing (that is, $\Theta = 0$), which is costless regardless of the value of $\bar{\Theta}$. Still, the null portfolio would exert some price effects, which are understood as the effects of leaving the market, or "undoing" the demand $\bar{\Theta}$.

[13] The approximation assures that in the definition of stability the second order condition is always satisfied. Otherwise, the approximation is without loss of generality.

Definition 3 A profile $\bar{M}$ of price impact matrices (for all traders) is *mutually consistent* given prices and trades $(\bar{P}, \bar{\Theta})$ if, for each individual i, there exist open neighborhoods $N^i(\bar{\Theta}^i)$ and $N^i(\bar{P})$, and a differentiable function $P^i : N^i(\bar{\Theta}^i) \to N^i(\bar{P})$, such that:

1. locally, P^i is the inverse demand faced by i:

$$\mathcal{P}^i(\tilde{\Theta}^i; M^{-i}) \cap N^i(\bar{P}) = \{P^i(\tilde{\Theta}^i)\} \tag{6}$$

   for every $\tilde{\Theta}^i \in N^i(\bar{\Theta}^i)$; and
2. matrix $\bar{M}^i$ is a correct estimation of the Jacobean of P^i at $\bar{\Theta}^i$: $\partial P^i(\bar{\Theta}^i) = \bar{M}^i$.

Note that consistency is imposed on the whole profile of individual matrices, and not matrix by matrix: the second and third conditions of the definition require that $\partial_{\tilde{\Theta}^i} P^i(\bar{\Theta}^i; M^{-i}) = M^i$ for every trader.[14] Intuitively, consistent price impact matrices reflect the price changes needed to clear the market for any small deviation $\tilde{\Theta}^i$ by trader i, given that other traders respond rationally to market prices in any subequilibrium. This means that each trader correctly estimates her price impact, to a first order, and incorporates this information into her individual decision problem. This contrasts with the competitive model, in which traders incorrectly assume that their price impacts are null;[15] put simply, traders in thin markets do not act as price takers, and behave as "slope takers" instead.

2.3 Equilibrium

The three principles of equilibrium in thin financial markets have now been introduced: equilibrium is a triple of prices, trades and perceptions of price impacts such that all markets clear; all traders are acting optimally, given the perception they hold of their price impacts; and these perceptions are accurate estimates of the actual price impacts exerted by the traders.

Definition 4 An *equilibrium in thin financial markets* for economy $\{u, e, R\}$ is a triple consisting of a vector of asset prices, a profile of portfolios and a profile of price impact matrices, $(\bar{P}, \bar{\Theta}, \bar{M})$, such that:

1. $\sum_i \bar{\Theta}^i = 0$;
2. for each trader i, trade $\bar{\Theta}^i$ is stable given $(\bar{P}, \bar{M}^i)$; and

[14] Strictly speaking, the intuition given before the definition is slightly weaker than the definition itself: we do not require that subequilibrium prices be unique (i.e., that $\mathcal{P}^i(\tilde{\Theta}^i; M^{-i})$ be singleton-valued), but only that they be determinate, or locally unique.

[15] The null price-impact beliefs assumed by the competitive model constitute an inconsistent profile M: if all traders but one believe that they exert no effect on prices, then the remaining trader faces, under our assumptions, a downward sloping demand function; if she is to have a correct estimate of the slope of the demand she faces, she cannot act as in the competitive model. However, competitive behavior appears as the limit of our model as markets deepen: let $(\bar{P}, \bar{\Theta})$ be the unique competitive equilibrium of economy $\{U, e\}$, and for each $n \in \mathbf{N}$, let (P_n, Θ_n, M_n) be an equilibrium for its n-fold replica, with the property that all M_n^i give no irrelevant impact; then, $(P_n, \Theta_n, M_n) \to (\bar{P}, \bar{\Theta}, 0)$, as shown by Weretka (2007b).

3. the profile of price impact matrices $\bar{M}$ is mutually consistent, given prices and trades $(\bar{P}, \bar{\Theta})$.

Importantly, note that at equilibrium all trades take place at prices $\bar{P}$. The following lemma provides a characterization of equilibrium for the case when there are no redundant assets in the market. For simplicity of notation, given an array of $I-1$ positive definite matrices, Δ, let $\mathcal{H}(\Delta)$ denote the harmonic average of the array.[16]

Lemma 1 *Suppose that R is nonsingular.*[17] *Then, $(\bar{P}, \bar{\Theta}, \bar{M})$ is an equilibrium in thin financial markets for economy $\{u, e, A\}$ if, and only if,*

1. $\sum_i \bar{\Theta}^i = 0$;
2. *for each trader i,* $P + M^i \bar{\Theta}^i = R^\top \partial u^i (e^i + R\bar{\Theta}^i)$;
3. *for each trader i,*

$$M^i = \frac{1}{I-1} \mathcal{H}((M^j - R^\top \partial^2 u^j (e^j + R\bar{\Theta}^j) R)_{j \neq i}). \tag{7}$$

Proof See Appendix A1. □

3 Arbitrage and irrelevant price impacts

The primary objective of this paper is to demonstrate that equilibrium in thin financial markets is not inconsistent with the standard no-arbitrage principle. Therefore, we now review the concept of no-arbitrage, introduce a weaker version of that principle, and also introduce an analogous condition, which we call the "no irrelevant impact" condition, for price impact matrices, and discuss its implication for economies with elementary securities.

3.1 No-arbitrage

A vector of asset prices P *allows no arbitrage opportunities* if $R\Theta > 0$ implies $P \cdot \Theta > 0$. It is well known that price vector P allows no-arbitrage opportunities if, and only if, there exists a vector of strictly positive "state prices," $p \in \mathbf{R}^S_{++}$, that determines the value of income in each state of the world: $P = R^\top p$. Consequently, the set of asset prices that allow no arbitrage opportunities is the convex, open, positive cone

$$\{P \in \mathbf{R}^A | \exists p \in \mathbf{R}^S_{++} : P = R^\top p\}. \tag{8}$$

A weaker condition of no-arbitrage would be the requirement that portfolios whose future return is zero in all states of the world should have zero cost at market prices.[18]

[16] That is, $\mathcal{H}((\Delta^j)_{j \in \mathcal{J}}) = (I-1)(\sum_{j \in \mathcal{J}} (\Delta^j)^{-1})^{-1}$.

[17] So that $A = S$ and $R_S = R$.

[18] In other words, the "law of one price" would apply in the asset markets, in the sense that doing and undoing some trade in commodities always has null cost.

That is, we will say that a vector of asset prices P *allows no weak arbitrage opportunities* if $R\Theta = 0$ implies $P \cdot \Theta = 0$. This weaker no-arbitrage condition amounts to a relaxation of the requirement that state prices be strictly positive, as in the following lemma.

Lemma 2 *The set of asset prices that allow no weak arbitrage opportunities is*

$$\{P \in \mathbf{R}^A | \exists p \in \mathbf{R}^S : P = R^\top p\}. \tag{9}$$

Proof See Appendix A1. □

If a trader who has monotone preferences and takes prices as given can find an optimal portfolio, it must be that the prices of the assets allow neither strong nor weak arbitrage opportunities; in particular, in a competitive equilibrium of the economy we are considering there can be no arbitrage opportunities. In the following sections we show that when markets are thin, only the weak notion of no-arbitrage is implied by optimization of a rational agent.[19] In order to claim the strong no-arbitrage condition, one also needs to invoke the market clearing condition.

Henceforth, we say that a pair of vectors of prices $(P, p) \in \mathbf{R}^A \times \mathbf{R}^S$ is R-*associated* if they satisfy $P = R^\top p$. By our assumption that markets are complete, it is immediate that (P, p) is R- associated only if $p = (R_S^\top)^{-1}(P_1, \ldots, P_S)^\top$. Consequently, R-association of prices defines a bijection between the linear subspace of all asset prices P satisfying the weak no-arbitrage condition, set (9), and $\mathbf{R}^S$. In the case of strong arbitrage, such bijection is between a convex cone, set (8), and $\mathbf{R}^S_{++}$.

If (P, p) is R-associated and P allows no arbitrage opportunities in the strong sense, then vector $(\sum_s p_s)^{-1} p \gg 0$ is a probability measure often referred to as the pricing kernel implied by P. Similarly, if P allows no weak arbitrage opportunities, then vector p can be interpreted as a (weak) pricing kernel implied by P. Note, however, that the weak pricing kernel does not have to be a probability measure since prices do not necessarily sum up to one and for some states they can even take negative values.

3.2 Price impact

A condition on price impact matrices that is analogous to the no-arbitrage principle on asset prices is the requirement that only portfolios with non-zero future payoff should affect prices. That is, we say that a price impact matrix M^i *gives no irrelevant impact* if $R\Theta^i = 0$ implies $M^i\Theta^i = 0$.[20] Technically, M^i gives no irrelevant impact if the kernel of matrix M^i contains the kernel of matrix R. It is immediate that, under this condition, portfolios with identical payoffs should have identical price effects. The next result provides a unique decomposition for symmetric matrices that give no irrelevant price impact; for any M^i define $m^i = (R_S^\top)^{-1} M_S^i (R_S)^{-1}$, where M_S^i is the $S \times S$ leading principal minor of matrix M^i.

[19] For this result, one also needs to assume the condition of no irrelevant impact, defined in the following section.

[20] Analogously to footnote 18, this condition can be seen as a "law of one price impact."

Lemma 3 *Let M^i be an $A \times A$, symmetric price impact matrix. If M^i gives no irrelevant impact, then*

1. *M^i can be written as $M^i = R^\top m^i R$; and*
2. *m^i is the only matrix that allows for such decomposition of M^i.*

Proof See Appendix A1. □

In the same vein as with prices, we say that a pair of an $A \times A$ matrix and an $S \times S$ matrix (M^i, m^i) is *R-associated* if $M^i = R^\top m^i R$. From Lemma 3, it follows that if M^i is symmetric and gives no irrelevant impact, then (M^i, m^i) is R-associated if, and only if, $m^i = (R_S^\top)^{-1} M_S^i (R_S)^{-1}$. Consequently, R-association defines a bijection between these sets of $A \times A$ and $S \times S$ matrices. The next two results show that the presence of one investor who trades optimally and whose trade has no irrelevant impact is sufficient for the weak no-arbitrage condition.

Proposition 1 *Suppose that price impact matrix $\bar{M}^i$ gives no irrelevant impact, and that there exists a portfolio $\bar{\Theta}^i$ that is stable for i given $(\bar{P}, \bar{M}^i)$. Then, prices $\bar{P}$ allow no weak arbitrage opportunities.*

Proof By stability of the individual's portfolio, it follows that

$$\bar{P} + \bar{M}^i \bar{\Theta}^i = R^\top \partial u^i (e^i + R\bar{\Theta}^i). \tag{10}$$

Multiplying (10) by any Θ^i such that $R\Theta^i = 0$ gives

$$\bar{P} \cdot \Theta^i = \partial u^i (e^i + R\bar{\Theta}^i)^\top R\Theta^i - (\bar{\Theta}^i)^\top \bar{M}^i \Theta = 0, \tag{11}$$

since $\bar{M}^i$ gives no irrelevant price impact. □

Intuitively, changes in the portfolio that do not affect payoffs do not alter consumption either, and, hence, do not change the marginal rate of substitution between assets. Since, at an optimum, the marginal rate of substitution equals marginal revenue, $\bar{P} + \bar{M}^i \bar{\Theta}^i$, the latter is not affected either. But, with no irrelevant impact, marginal revenue is constant only if prices are orthogonal to zero-payoff portfolios.

Corollary 1 *If $(\bar{P}, \bar{\Theta}, \bar{M})$ is an equilibrium in thin financial markets for $\{u, e, R\}$, and if for at least one trader her price impact matrix gives no irrelevant impact, then $\bar{P}$ allows no weak arbitrage opportunities, and there exists a (weak) pricing kernel, $p \in \mathbf{R}^S$ such that $P = R^\top p$.*

Proof This follows from Proposition 1 and Lemma 2. □

The next result now demonstrates that, in equilibrium, if all traders but one estimate that they have no irrelevant price impact, then that remaining trader will automatically have no irrelevant impact.

Proposition 2 *Let $(\bar{P}, \bar{\Theta}, \bar{M})$ be an equilibrium in thin financial markets for economy $\{u, e, R\}$. If for all traders $j \neq i$ their price impact matrices give no irrelevant price impact, then matrix $\bar{M}^i$ gives no irrelevant impact either.*

Before the proof of the proposition, we introduce a lemma whose proof we again defer to Appendix A1.

Lemma 4 *Return-equivalent deviations trigger the same subequilibrium prices: Fix an individual i and a profile of price impact matrices for all traders but i, $\bar{M}^{-i} = (\bar{M}^j)_{j\neq i}$, and suppose that each matrix $\bar{M}^j$ gives no irrelevant price impact. Suppose that portfolios $\tilde{\Theta}^i$ and $\hat{\Theta}^i$ are such that $R\tilde{\Theta}^i = R\hat{\Theta}^i$. If $(\tilde{P}, \tilde{\Theta}^{-i})$ is a subequilibrium triggered by trade $\tilde{\Theta}^i$, then*

$$\left(\tilde{P}, (\tilde{\Theta}^j + (I-1)^{-1}(\tilde{\Theta}^i - \hat{\Theta}^i))_{j\neq i}\right) \tag{12}$$

is a subequilibrium triggered by trade $\hat{\Theta}^i$.

Proof of Proposition 2 Let Θ^i be a portfolio such that $R\Theta^i = 0$. By definition of equilibrium, there exists a local inverse demand function for individual i, P^i. Define the function $\varphi(\lambda) = P^i(\bar{\Theta}^i + \lambda\Theta^i)$, for real numbers λ close enough to 0. By Lemma 4, it follows that $(\bar{P}(\bar{\Theta}^j - (I-1)^{-1}\lambda\Theta^i)_{j\neq i})$ is a subequilibrium triggered by trade $\lambda\Theta^i$, so it is immediate that $\varphi'(\lambda) = 0$. By construction, $\varphi'(\bar{\Theta}^i)\Theta^i = \bar{M}^i\Theta^i$. □

We restrict our attention to equilibria in which all price impact matrices give no irrelevant impact. In the following sections we argue that such equilibria exist and are determinate. We conjecture that equilibria in which traders have irrelevant impact do not exist.

4 Pricing kernel in thin complete markets

4.1 Elementary securities

Since we are considering only the case of complete financial markets, we can obtain a canonical auxiliary representation of the economy by endowing with a complete set of elementary securities, in lieu of the set of securities R. That is, the representation of economy $\{u, e, R\}$ in elementary securities is economy $\{u, e, \mathbf{I}_S\}$. For notational simplicity, we use lower-case characters for the corresponding (endogenous) variables in the representation in elementary securities:

Definition 5 Given preferences and endowments $\{u, e\}$, a triple consisting of a vector of state prices, a profile of revenue transfers across states of the world and a profile of state-price impact matrices, $(\bar{p}, \bar{\theta}, \bar{m})$ is an *equilibrium in elementary securities* for $\{u, e\}$ if it is an equilibrium in thin financial markets for economy $\{u, e, \mathbf{I}_S\}$.

The following result says that, under our assumptions, equilibria in elementary securities exist and are (generically) locally unique.

Proposition 3 *Suppose there are at least three traders in the market. Then,*

1. *For any profile of preferences and endowments, $\{u, e\}$, there exists an equilibrium in elementary securities $(\bar{p}, \bar{\theta}, \bar{m})$ such that all price impact matrices $\bar{m}^i$ are diagonal and positive definite.*

2. *If, moreover, all preferences are three times continuously differentiable, then, generically in the space of preferences and endowments, all equilibria in elementary securities with diagonal, positive-definite price impact matrices are locally unique (i.e., isolated from one another).*[21]

Proof These results follow, respectively, from Theorems 1 and 2 in Weretka (2007a). For the sake of simplicity and completeness, Appendix A2 specializes the proofs to our case. □

It is natural to ask whether in the equilibria resulting from Proposition 3 there are arbitrage opportunities and irrelevant price impacts. Observe that in the economy with elementary securities the only portfolio that gives zero return in all states is a zero portfolio, and obviously both the cost of such portfolio and its price impact are zero, regardless of the specific prices and price impact matrices. Therefore, equilibrium prices do not allow for weak arbitrage opportunities and there is no irrelevant price impact in the economy with elementary securities. The question of whether the no-arbitrage condition holds in the strong sense (so that a strong pricing kernel exists) is addressed in the next proposition.

Proposition 4 *Let* $(\bar{p}, \bar{\theta}, \bar{m})$ *be an equilibrium in elementary securities for* $\{u, e\}$*, with diagonal, positive definite price impact matrices* $\bar{m}^i$*. Then, prices* $\bar{p}$ *allow no-arbitrage opportunities:* $\bar{p} \gg 0$

Proof For every trader i, it must be true that $\bar{p} = \partial u^i(e^i + \bar{\theta}^i) - \bar{m}^i \bar{\theta}^i$. For asset (state) s, since $\bar{m}^i$ is diagonal, the latter means that $\bar{p}_s = \partial u^i_s(e^i_s + \bar{\theta}^i_s) - \bar{m}^i_{s,s}\bar{\theta}^i_s$. By market clearing, there must exist i for whom $\bar{\theta}^i_s \leq 0$, which means, by monotonicity of u^i_s and positive definiteness of $\bar{m}^i$, that $\bar{p}_s > 0$. □

Critically, the implication of Proposition 4 does not follow simply from the existence of one rational trader with monotone preferences; rather, it follows from the facts that all traders are acting optimally according to their von Neumann–Morgenstern utility functions, which are separable, and that all markets clear.

4.2 Equivalence of the representations

We say that a pair of portfolios $(\Theta^i, \theta^i) \in \mathbf{R}^A \times \mathbf{R}^S$ is *R-associated* if the two portfolios are payoff-equivalent, $R\Theta^i = \theta^i$. By market completeness, for every θ^i there is at least one Θ^i such that (Θ^i, θ^i) is R-associated, for example

$$\Theta^i = \begin{pmatrix} R_S^{-1}\theta^i \\ 0 \end{pmatrix}. \tag{13}$$

When there are no redundant assets, such Θ^i is unique.

[21] The sense in which this result holds for a "typical economy" is established in detail in the proof.

Furthermore, we say that a triple of a vector of prices, a profile of portfolios and a profile of price impact matrices in market R, and a corresponding triple in elementary securities are *R-associated* if so are each of their components: (P, Θ, M) and (p, θ, m) are R-associated if:

1. the pair of prices (P, p) is R-associated: $P = R^{\top} p$;
2. for each i, the pair of portfolios (Θ^i, θ^i) is R-associated: $R\Theta^i = \theta^i$; and
3. for each i, the pair of price impact matrices (M^i, m^i) is R-associated: $M^i = R^{\top} m^i R$.

The next lemma establishes the connection between the equilibria in economies with arbitrary assets and their counterparts with elementary securities.

Lemma 5 *Suppose that* $(\bar{P}, \bar{\Theta}, \bar{M})$ *and* $(\bar{p}, \bar{\theta}, \bar{m})$ *are R-associated. Then,*

1. *if* $(\bar{p}, \bar{\theta}, \bar{m})$ *is an equilibrium in elementary securities for* $\{u, e\}$ *and* $\sum_i \bar{\Theta}^i = 0$, *then* $(\bar{P}, \bar{\Theta}, \bar{M})$ *is an equilibrium in thin financial markets for* $\{u, e, R\}$;
2. *if* $(\bar{P}, \bar{\Theta}, \bar{M})$ *is an equilibrium in thin financial markets for* $\{u, e, R\}$, *then* $(\bar{p}, \bar{\theta}, \bar{m})$ *is an equilibrium in elementary securities for* $\{u, e\}$.

Proof See Appendix A1. □

In the following sections, we explore the theoretical and practical implications of this result.

4.3 Equilibria with no-arbitrage

Suppose that trader i has chosen a portfolio Θ^i, even though an arbitrage opportunity Θ exists in the market. If i were competitive, portfolio $\Theta^i + \lambda\Theta$, for any $\lambda > 0$, would give her more utility. This need not be the case, though, when i is noncompetitve, for her demand for portfolio Θ, the arbitrage opportunity, may exert 'detrimental' effects of the price of Θ itself, and on the cost of portfolio Θ^i; while the first effect is of second order, and hence vanishes for small λ, the second effect is of first order.

Proposition 4 has given a condition under which no-arbitrage remains a necessary condition of equilibria under noncompetitive behavior, for the case of a full set of elementary securities. The following result is the most important one in the paper: it shows that in economies with complete markets and separable preferences, noncompetitive behavior is not incompatible with the existence of a pricing kernel.

Theorem 1 *Suppose that there are at least three traders in the market. Then,*

1. *For any economy* $\{u, e, R\}$, *there exists an equilibrium* $(\bar{P}, \bar{\Theta}, \bar{M})$ *satisfying that*
   (a) *all price impact matrices* $\bar{M}^i$ *are symmetric, positive definite and give no irrelevant price impact; and*
   (b) *prices* $\bar{P}$ *allow no strong (and, hence, no weak) arbitrage opportunities, and a classical pricing kernel exists.*
2. *If, moreover, all individual preferences are three times continuously differentiable, then, for a generic profile of preferences and endowments* $\{u, e\}$, *and for any asset structure* R, *equilibria satisfying conditions (a) and (b) above are locally*

unique, at least up to the payoff equivalence of the portfolios: generically in $\{u, e\}$*, every equilibrium* $(\bar{P}, \bar{\Theta}, \bar{M})$ *of* $\{u, e, R\}$ *has an open neighborhood such that if* $(\tilde{P}, tilde\Theta, \tilde{M})$ *is an equilibrium for the same economy and lies in that open neighborhood, then* $\tilde{P} = \bar{P}$, $\tilde{M} = \bar{M}$ *and* $R\tilde{\Theta}^i = R\bar{\Theta}^i$ *for every trader* i*.*

Proof Part 1 follows from part 1 of Proposition 3: Given preferences and endowments $\{u, e\}$, it follows from the proposition that there exists an equilibrium in elementary securities $(\bar{p}, \bar{\theta}, \bar{m})$, and that in this equilibrium all price impact matrices $\bar{m}^i$ are diagonal. Define $\bar{P} = R^\top \bar{p}$ and, for all trader i, let

$$\bar{\Theta}^i = \begin{pmatrix} R_S^{-1}\bar{\theta}^i \\ 0 \end{pmatrix} \tag{14}$$

and $\bar{M}^i = R^\top \bar{m}^i R$. By construction, $(\bar{P}, \bar{\Theta}, \bar{M})$ and $(\bar{p}, \bar{\theta}, \bar{m})$ are R-associated and $\sum_i \bar{\Theta}^i = 0$. It follows from part 1 of Proposition 5 that $(\bar{P}, \bar{\Theta}, \bar{M})$ is an equilibrium in thin financial markets for $\{u, e, R\}$. By construction, every price impact matrix $\bar{M}^i$ gives no irrelevant price impact, and hence, it follows from Proposition 1 that P allows no weak arbitrage opportunities. Moreover, since every matrix $\bar{m}^i$ is diagonal, it follows from part 1 of Proposition 4 that $\bar{p} \gg 0$, and, hence, that $\bar{P} = R^\top \bar{p}$ allows no strong arbitrage opportunities either.

For part 2, take any profile of preferences and endowments, $\{u, e\}$, in the generic set given by part 2 of Proposition 3, and fix any R satisfying the assumptions imposed in the paper. Suppose, by way of contradiction, that there exist an equilibrium $(\bar{P}, \bar{\Theta}, \bar{M})$ and a sequence of equilibria $(P_n, \Theta_n, M_n)_{n=1}^\infty$, all of them satisfying conditions (a) and (b), such that

1. for every n, $(P_n, (R\Theta_n^i)_{i=1}^I, M_n) \neq (\bar{P}, (R\bar{\Theta}^i)_{i=1}^I, \bar{M})$; and
2. sequence $(P_n, \Theta_n, M_n)_{n=1}^\infty$ converges to $(\bar{P}, \bar{\Theta}, \bar{M})$.

Define $\bar{p}$ and p_n to be the unique state-price vectors R-associated, respectively, to $\bar{P}$ and P_n. Similarly, let $\bar{m}$ and m_n be the unique profiles of price impact matrices that are R-associated, respectively, to $\bar{M}$ and M_n. By part 2 of Lemma 5, it follows that $(\bar{p}, (R\bar{\Theta}^i)_{i=1}^I, \bar{m})$ and all $(p_n, (R\Theta_n^i)_{i=1}^I, m_n)$ are equilibria in elementary securities for $\{u, e\}$. Since R-association of prices and price impact matrices both define bijections, it follows from the first property above that for every n, $(p_n, (R\Theta_n^i)_{i=1}^I, m_n) \neq (\bar{p}, (R\bar{\Theta}^i)_{i=1}^I, \bar{m})$. By the second property above, $(p_n, (R\Theta_n^i)_{i=1}^I, m_n)$ converges to the limit $(\bar{p}, (R\bar{\Theta}^i)_{i=1}^I, \bar{m})$, but this is impossible since all equilibria in elementary securities for $\{u, e\}$ are locally unique. □

5 The fundamental theorem of asset pricing

The following result is an extension of the fundamental theorem of asset pricing. It says the introduction of an asset whose payoff can be replicated by a combination of existing assets need not perturb non-competitive equilibria: the new asset is priced at the cost of the portfolio that replicates it, and it is not traded. What is new, though, is that the same principle by which the asset is priced applies to the price impacts it exerts and receives.

Theorem 2 *Suppose that (P, Θ, M) is an equilibrium in thin financial markets for economy $\{u, e, R\}$, and that all price impact matrices M^i give no irrelevant price impact. Let ρ be a collection of new assets, taken as an $S \times \tilde{A}$ matrix, let $\tilde{R} = (R, \rho)$ be a new asset structure, and let*

$$\Theta_\rho = \begin{pmatrix} R_S^{-1}\rho \\ 0 \end{pmatrix} \tag{15}$$

be a collection of portfolios that replicate the assets in ρ.[22] *Then, an equilibrium for economy $\{u, e, \tilde{R}\}$ is given by $(\tilde{P}, \tilde{\Theta}, \tilde{M})$ with prices*

$$\tilde{P} = \begin{pmatrix} P \\ \Theta_\rho^\top P^\top \end{pmatrix}, \tag{16}$$

individual portfolios $\tilde{\Theta}^i = ((\Theta^i)^\top, 0)^\top$, and individual price impact matrices

$$\tilde{M}^i = \begin{pmatrix} M^i & M^i\Theta_\rho \\ \Theta_\rho^\top M^i & \Theta_\rho^\top M^i \Theta_\rho \end{pmatrix}. \tag{17}$$

Proof By Corollary 1, there exists $p \in \mathbf{R}^S$ such that $R^\top p = P$. By construction, then, $p = (R_S^\top)^{-1}(P_1, \ldots, P_S)^\top$. Define also, for each $i, \theta^i = R\Theta^i$ and $m^i = (R_S^\top)^{-1} M_S^i R_S^{-1}$. By Lemma 3, (P, Θ, M) and (p, θ, m) are R-associated, and, in particular,

$$M^i = R^\top m^i R = \begin{pmatrix} M_S^i & M_S^i R_S^{-1} R_{A-S} \\ R_{A-S}^\top (R_S^{-1})^\top M_S^i & R_{A-S}^\top (R_S^{-1})^\top \quad M_S^i R_S^{-1} R_{A-S} \end{pmatrix}. \tag{18}$$

It follows from part 2 of Proposition 5 that (p, θ, m) is an equilibrium in elementary securities for $\{u, e\}$. Also by construction, prices $(\tilde{P}, p)$ and portfolios $(\tilde{\Theta}, \theta)$ are $\tilde{R}$-associated. Moreover, for each trader i,

$$\tilde{R}^\top m^i \tilde{R} = \begin{pmatrix} M^i & R^\top m^i \rho \\ \rho^\top m^i R & \rho^\top m^i \rho \end{pmatrix}. \tag{19}$$

By direct computation,

$$R^\top m^i \rho = \begin{pmatrix} M_S^i R_S^{-1} \rho \\ R_{A-S}^\top (R_S^\top)^{-1} \quad M_S^i R_S^{-1} \rho \end{pmatrix} = M^i \Theta_\rho, \tag{20}$$

and $\rho^\top m^i \rho = \Theta_\rho^\top M^i \Theta_\rho$, so $\tilde{R}^\top m^i \tilde{R} = \tilde{M}^i$. It follows that $(\tilde{M}, m)$ is $\tilde{R}$-associated, and the result then follows from part 1 of Proposition 5. □

[22] Notice that Θ_ρ is an $A \times \tilde{A}$ matrix, and not just a column vector. The ath column in this matrix replicates the ath payoff vector in ρ.

Corollary 1 and Proposition 2 can be used to write the equilibrium prices for the economy endowed with asset structure $\tilde{R}$ as

$$\tilde{P} = \tilde{R}^{\mathsf{T}} p \begin{pmatrix} P \\ \rho^{\mathsf{T}} p \end{pmatrix}. \tag{21}$$

for some $p \in \mathbf{R}^S$. It is in this sense that vector p obtained from corollary is a pricing kernel: the prices of all assets, including derivatives that can be replicated from existing assets, can be expressed as the value of their returns at state prices p. Crucially, Proposition 1 established the existence of equilibria in which p is a pricing kernel in the classical sense ($p \gg 0$) and can be used to define a probability measure over states of the world.

Note also that the computation of price impact matrices can be obtained via their analogous in elementary securities as well: this is the intuition of Eq. (19). In this sense, one can view the price impact matrices in elementary securities as "price impact kernels."

6 An example

We now use a "coconut" example to illustrate numerical differences between equilibria in competitive economies and equilibria in thin financial markets. We choose the values of parameters to illustrate the mechanisms that operate in thin financial markets. We demonstrate that such mechanisms potentially may help to explain some of the asset pricing puzzles. Finally we show how Theorem 2 can be applied to price derivatives in thin markets.

Consider an island where the only production technology available is coconut palms. There are three equally likely states of the world, $s = 1, 2, 3$: in state 1 there is a drought in the north of the island and the palms there are not productive; in state 2, the drought is in the south of the island; in state 3, there is no drought anywhere in the island, and in this case each palm produces two coconuts.

There are two types of individual, $t = 1, 2$, and there are two (identical) individuals of each type. Type-1 individuals live in the north, so they have wealth only when that region of the island is productive: they are endowed with $e^1 = (0, 2, 2)$. Type-2 individuals live in the south, and, analogously, they have $e^2 = (2, 0, 2)$. All four individuals have identical preferences,

$$u(c) = c_0 + \frac{1}{3} \sum_s \ln(c_s). \tag{22}$$

There are three securities that can be traded in this economy: a riskless bond, $R^1 = (1, 1, 1)^{\mathsf{T}}$, a share in a coconut palm from the south, $R^2 = (0, 2, 2)^{\mathsf{T}}$, and a share in a coconut palm from the north, $R^3 = (2, 0, 2)^{\mathsf{T}}$.

6.1 Equilibrium in thin financial markets

If all traders ignore their price effects and behave competitively, the equilibrium in these financial markets is at prices $\tilde{P} = (5/6, 1, 1)^{\mathsf{T}}$, with both type-1 individuals buying a portfolio $\tilde{\Theta}^1 = (0, -1/2, 1/2)^\top$ and both type-2 individuals selling that portfolio.

6.1.1 Elementary securities

For simplicity of computation, we can solve first for an equilibrium in elementary securities, and then use Lemma 5 to obtain one in the original thin financial markets. By numerical computation, an equilibrium in elementary securities is as follows:[23]

1. prices are $\bar{p} = (0.3920, 0.3920, 0.1667)^\top$;
2. both type-1 individuals buy a portfolio $\bar{\theta}^1 = (0.6647, -0.6647, 0)$ and estimate a price impact matrix

$$\bar{m}^1 = \begin{pmatrix} 0.1646 & 0 & 0 \\ 0 & 0.2142 & 0 \\ 0 & 0 & 0.0417 \end{pmatrix}; \tag{23}$$

3. each type-2 individual sells the portfolio purchased by a type-1 individual, $\bar{\theta}^2 = -\bar{\theta}^1$, and they both estimate a price impact matrix

$$\bar{m}^2 = \begin{pmatrix} 0.2142 & 0 & 0 \\ 0 & 0.1646 & 0 \\ 0 & 0 & 0.0417 \end{pmatrix}. \tag{24}$$

6.1.2 Equilibrium

We can now use Lemma 5 to compute the equilibrium with asset structure $R = (R^1, R^2, R^3)$:

1. prices are given by

$$\bar{P} = R^\top \bar{p} = (0.9507, 1.1173, 1.1173)^\top; \tag{25}$$

2. each individual of type 1 buys a portfolio

$$\bar{\Theta}^1 = R^{-1}\bar{\theta}^1 = \begin{pmatrix} 0.0000 \\ -0.3324 \\ 0.3324 \end{pmatrix}, \tag{26}$$

   which is sold by a type-2 individual: $\bar{\Theta}^2 = -\bar{\Theta}^1$; and

[23] This equilibrium can be compared with the competitive equilibrium of this economy in the case of a full set or elementary securities: prices would be $\tilde{p} = (1/3, 1/3, 1/6)^\top$, type-1 individuals would buy $\tilde{\theta}^1 = (1, -1, 0)^\top$, which would be sold by type-2 individuals.

3. each individual of type 1 estimates that her price impacts are

$$\bar{M}^1 = R^{\mathsf{T}} \bar{m}^1 R = \begin{pmatrix} 0.4205 & 0.5117 & 0.4126 \\ 0.5117 & 1.0233 & 0.1667 \\ 0.4126 & 0.1667 & 0.8252 \end{pmatrix}, \tag{27}$$

while each individual of type 2 estimates

$$\bar{M}^2 = R^{\mathsf{T}} \bar{m}^2 R = \begin{pmatrix} 0.4205 & 0.4126 & 0.5117 \\ 0.4126 & 0.8252 & 0.1667 \\ 0.5117 & 0.1667 & 1.0233 \end{pmatrix}. \tag{28}$$

One can readily verify that these values satisfy all the conditions of Lemma 1, and, hence, constitute an equilibrium for $\{u, e, R\}$.

This application illustrates the use of our no-arbitrage results. As is learnt from the literature on competitive asset pricing, the equilibrium prices of elementary securities constitute a pricing kernel that allows us to find the equilibrium prices under other complete financial securities. Our results show that this formulae are still applicable under non-competitiveness, and that an analogous exercise can be applied to construct the price impact matrices for arbitrary asset structures on the basis of those obtained at equilibrium under elementary securities.[24]

6.2 Price biases

6.2.1 Risk-free interest rates

The risk-free interest rate puzzle was first observed by Weil (1989). In short, in the competitive theory of asset pricing, the risk-averse agents prefer flat consumption over time. They are willing to postpone consumption only if compensated by a high interest rate. Given the standard preferences, the empirically observed growth rate of per capita consumption equal to 2% cannot be reconciled with the risk-free interest rate of only 1%.

Notice that in the economy of our example, the competitive risk-free interest rate is $\tilde{r} = 1/\tilde{P}_1 - 1 = 0.2$. Importantly, notice that when these agents recognize their price impacts the risk-free interest rate is lower: it is $\bar{r} = 1/\bar{P}_1 - 1 = 0.0519$. Importantly, it turns out that this result is more general than just the current example. It follows from the Fundamental Theorem of Asset Pricing that, with a pricing kernel, the return to a riskless asset, $1 + r$, can be computed as the inverse of the sum of all state prices. Suppose now that all individuals have identical CRRA Bernoulli utility functions.[25]

[24] Of course, while the pricing kernel applies to all the traders, each one of them will have an individual "price impact kernel".

[25] Namely, if for all i,

$$u_s^i(c_s^i) = \frac{(c_s^i)^{1-\phi_s}}{1-\phi_s}, \tag{29}$$

Since the third derivative of a CRRA utility function is positive, the assumptions of Proposition 4 in Weretka (2007a) hold. Then, state-by-state, the equilibrium price of an elementary security is above its competitive counterpart and therefore the risk-free interest rate is lower than in the corresponding competitive market. We formally have, then, the following proposition.

Proposition 5 *Consider an economy* $\{U, e, R\}$*, where all individuals have identical CRRA Bernoulli utility functions. In any equilibrium satisfying the two properties stated in part 1 on Theorem 1, the risk-free interest rate is always below the rate that would arise in a competitive economy.*

It follows that when financial markets are thin, using the competitive model to explain the empirical returns will lead to a risk-free interest rate puzzle. This is true even if the specified CRRA preferences have a "reasonable" risk-aversion parameter.

In a competitive model with identical quasilinear preferences consumption of each agent is the same and equal to a per capita consumption. In thin markets, however traders do not perfectly hedge their idiosyncratic risk, and hence equilibrium consumption is not degenerated. With convex marginal utility,[26] in each state the same per capita consumption is associated with a higher average marginal utility across all traders. This enhances the price of an elementary security. Consequently the interest rate falls.

The reason why, given a low interest rate, agents do hold bonds rather than consume more in period zero is that with a convex marginal utility and partial hedge of idiosyncratic risk, an agent's marginal value of a dollar is disproportionally high in bad states; the riskless asset pays in all states a constant amount and therefore is an attractive insurance instrument.

6.2.2 Equity premia

Another anomaly robustly documented in the empirical literature is the equity premium puzzle. In our example, if trade were competitive the equity premium would be, for asset 2,

$$\frac{E(R^2)}{\tilde{P}_2} - \tilde{r} = 0.1333. \tag{30}$$

In the equilibrium in thin markets, the same return is traded at a higher premium

$$\frac{E(R^2)}{\bar{P}_2} - \bar{r} = 0.1414. \tag{31}$$

Therefore in our example the equity premium is significantly above the one obtained in the competitive setting. Intuitively, price impacts of the traders are the highest for

footnote 25 continued
where $\phi_s > 0$ is the coefficient of risk aversion; it is well known that in the case $\phi_s = 1$ one has that $u_s^i(c_s^i) = \ln(c_s^i)$.

[26] The third derivative of a CRRA utility function is positive, which implies that in each state a marginal utility is convex.

elementary securities corresponding to “bad” states of the economy, and for such states the price biases are strongest.

6.3 Arbitrage pricing in practice

To the existing assets, a bond and two stocks, we add call options shares of the palms, with strike prices equal to 1 in both cases. These assets can be written as

$$R^4 = \begin{pmatrix} \max\{R_1^2 - 1, 0\} \\ \max\{R_2^2 - 1, 0\} \\ \max\{R_3^2 - 1, 0\} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{32}$$

and

$$R^5 = \begin{pmatrix} \max\{R_1^3 - 1, 0\} \\ \max\{R_2^3 - 1, 0\} \\ \max\{R_3^3 - 1, 0\} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{33}$$

Let $\rho = (R^4, R^5)$, and notice that these claims are reproduced by

$$\Theta_\rho = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \tag{34}$$

By Theorem 2, we can obtain an equilibrium where prices of the existing (primitive) assets remain the same, all traders constitute the same portfolios as before (that is, they do not trade the options), the options are priced at $\bar{P}\Theta_\rho = (0.5587, 0.5587)$, and the price impact matrices are as follows: for each type-1 individual the matrix is

$$\begin{pmatrix} \bar{M}^1 & \bar{M}^1\Theta_\rho \\ \Theta_\rho^\top \bar{M}^1 & \Theta_\rho^\top \bar{M}^1\Theta_\rho \end{pmatrix} = \begin{pmatrix} 0.4205 & 0.5117 & 0.4126 & 0.2558 & 0.2063 \\ 0.5117 & 1.0233 & 0.1667 & 0.5117 & 0.0833 \\ 0.4126 & 0.1667 & 0.8252 & 0.0833 & 0.4126 \\ 0.2558 & 0.5117 & 0.0833 & 0.2558 & 0.0417 \\ 0.2063 & 0.0833 & 0.4126 & 0.0417 & 0.2063 \end{pmatrix}, \tag{35}$$

while for each type-2 individual it is

$$\begin{pmatrix} \bar{M}^2 & \bar{M}^2\Theta_\rho \\ \Theta_\rho^\top \bar{M}^2 & \Theta_\rho^\top \bar{M}^2\Theta_\rho \end{pmatrix} = \begin{pmatrix} 0.4205 & 0.4126 & 0.5117 & 0.2063 & 0.2558 \\ 0.4126 & 0.8252 & 0.1667 & 0.4126 & 0.0833 \\ 0.5117 & 0.1667 & 1.0233 & 0.0833 & 0.5117 \\ 0.2063 & 0.4126 & 0.0833 & 0.2063 & 0.0417 \\ 0.2558 & 0.0833 & 0.5117 & 0.0417 & 0.2558 \end{pmatrix}. \tag{36}$$

These results are, again, applications of our extension of the fundamental theorem of asset pricing; they illustrate how the idea of no arbitrage pricing can be extended no our non-competitive setting.

7 Strategic market games

An alternative model of non-competitive interaction in markets is given by the concept of Nash equilibrium in strategic market games, as defined by Shapley and Shubik (1977).[27] In such a setting, Koutsougeras (2003) and Koutsougeras and Papadapoulos (2004) prove that arbitrage opportunities may exist at equilibrium prices.[28] The purpose of this section is to compare the definitions of equilibrium used in these two papers with the one used here, in order to understand the reason for the difference in their implications. We first cast our definition of equilibrium in thin markets in the setting of a strategic market game, and later express the definition of Nash equilibrium in such a game in the language used in the rest of the paper. For the sake of simplicity, we do this in the context of trade in elementary securities only, which provides us with a simpler notation.

7.1 Conjectural equilibrium in games

Before we can express the concept of equilibrium in thin markets in the language of strategic market games, we specialize a series of definitions to the case of a suitable class of games. Consider a set of players who are denoted by $i = 1, \ldots, I$. For each player i, let $\mathbf{A}^i \subseteq \mathbf{R}^K$ be the set of actions, and let $U^i : \mathbf{A} \to \mathbf{R}$ denote the preferences, where $\mathbf{A} = \times_i \mathbf{A}^i$.

Suppose that the game is subject to the following aggregation structure: let $\Pi \subseteq \mathbf{R}^S$ be a set of aggregate states, let $\boldsymbol{\pi} : \mathbf{A} \to \Pi$ be a smooth aggregator function, and suppose that for every player i, preferences satisfy that $U^i(\alpha) = v^i(\alpha^i, \boldsymbol{\pi}(\alpha))$, for some smooth function $v^i : \mathbf{A}^i \times \Pi \to \mathbf{R}$. Every concept below should be taken as relative to the aggregator $\boldsymbol{\pi}$.

7.1.1 Conjectures

A conjecture for individual i is a function $\pi^i : \mathbf{A}^i \times \mathbf{A} \to \Pi$. This function is understood as the individual's belief aggregate state she can induce from a given initial profile of actions: she believes that if the status quo is $\bar{\alpha}$, then she can induce state $\pi^i(\alpha^i, \bar{\alpha})$ by playing α^i.

If a conjecture has the property that, for every $\bar{\alpha}^i \in \mathbf{A}^i$ it is true that $\pi^i(\cdot, (\bar{\alpha}^i, \alpha^{\neg i})) = \pi^i(\cdot, (\bar{\alpha}^i, \bar{\alpha}^{\neg i}))$ whenever $\alpha^{\neg i}, \bar{\alpha}^{\neg i} \in \mathbf{A}^{\neg i}$ are such that $\boldsymbol{\pi}(\bar{\alpha}^i, \alpha^{\neg i}) = \boldsymbol{\pi}(\bar{\alpha}^i, \bar{\alpha}^{\neg i})$, then the conjecture is said to be *coarse*: in this case, the individual only distinguishes the status quo up to her own action and the aggregate state of the world. In the case of a coarse conjecture, we may simply write $\pi^i : \mathbf{A}^i \times \mathbf{A}^i \times \Pi \to \Pi$.

[27] See Giraud (2003) for a survey of this literature.

[28] In the economy presented in the previous section, though, this does not occur, at least for symmetric Nash equilibrium; we will come back to this point after the necessary notation has been introduced.

A coarse conjecture π^i is *variational* if there exists a function $\delta^i : \mathbf{A}^i \rightarrow \mathbf{R}^{K \times S}$ such that

$$\pi^i(\alpha^i, \bar{\alpha}^i, \mathbf{p}) = \mathbf{p} + \int_{\bar{\alpha}^i}^{\alpha^i} \delta^i(a) \mathrm{d}a \tag{37}$$

for every $(\alpha^i, \bar{\alpha}^i, \mathbf{p})$. If, furthermore, function δ^i is constant, we say that the conjecture is *simple*. In the case of a variational conjecture, it suffices that we identify function δ^i, which, in turn, we can take to be simply a $K \times S$ matrix when the conjecture is simple.

7.1.2 Conjectural equilibrium

Given a profile of actions $\bar{\alpha}$ and a conjecture π^i, define the set

$$B^i(\bar{\alpha}, \pi^i) := \operatorname{argmax}_{\alpha^i} v^i(\alpha^i, \pi^i(\alpha^i, \bar{\alpha})). \tag{38}$$

By construction, if the conjecture is coarse we can simply write this set as $B^i(\bar{\alpha}^i, \bar{\mathbf{p}}, \pi^i)$, with $\bar{\mathbf{p}} = \boldsymbol{\pi}(\bar{\alpha})$.

Any action $\bar{\alpha}^i$ such that $\bar{\alpha}^i \in B^i((\bar{\alpha}^i, \bar{\alpha}^{\neg i}), \pi^i)$ is *stable* for individual i, given the actions of others and her conjecture $(\alpha^{\neg i}, \pi^i)$. A *conjectural equilibrium with respect to the profile of conjectures* π is a profile of actions $\bar{\alpha}$ such no player would want to deviate, given her conjecture: $\bar{\alpha}^i \in B^i((\bar{\alpha}^i, \bar{\alpha}^{\neg i}), \pi^i)$ for all i.

The following lemma restates the definition of Nash equilibrium in the context of variational conjectures.[29]

Lemma 6 *The profile of actions $\bar{\alpha}$ is a Nash equilibrium of the game if, and only if, it is a conjectural equilibrium with respect to the profile of variational conjectures given by*

$$\alpha^i \mapsto \delta^i(\alpha^i) = \partial_{\alpha^i} \boldsymbol{\pi}(\alpha^i, \bar{\alpha}^{\neg i}), \tag{39}$$

for all i.

Proof See Appendix A1. □

This result simply says that, for a given Nash equilibrium, all that matters is the player's conjectured effect given the equilibrium actions of others.[30]

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[29] A *Nash equilibrium* is a conjectural equilibrium with respect to the profile of conjectures $((\alpha^i, \bar{\alpha}) \mapsto \boldsymbol{\pi}(\alpha^i, \bar{\alpha}^{\neg i}))_{i \in I}$.

[30] Importantly, note that these variational conjectures are constructed for each equilibrium, so it need not be possible to construct variational conjectures that will capture *all* the Nash equilibria of a game.

7.1.3 Consistent conjectural equilibrium

For each player i, given an action α^i and a profile of conjectures for players other than i, $\pi^{\neg i}$, let $\Pi^i(\alpha^i, \pi^{\neg i})$ be the projection into Π of the set

$$\{(p, \alpha^{\neg i}) \in \Pi \times A^{\neg i} : \boldsymbol{\pi}(\alpha^i, \alpha^{\neg i}) = p \text{ and } \alpha^j \in B^j((\alpha^1, \ldots, \alpha^I), \pi^j) \text{ for all } j \neq i\}. \tag{40}$$

Set $\Pi^i(\alpha^i, \pi^{\neg i})$ collects the aggregate states that individual i could induce by playing action α^i, if every other player acts optimally according to her conjecture.

A profile of conjectures π is *perfect* if for every individual i, and for every profile of actions of her opponents, $\alpha^{\neg i}$, one has that $\Pi^i(\alpha^i, \pi^{\neg i}) = \{\pi^i(\alpha^i, \alpha^{\neg i})\}$ for all of her actions. The profile is *consistent* if we simply require that $\pi^i(\alpha^i, \alpha^{\neg i}) \in \Pi^i(\alpha^i, \pi^{\neg i})$.

A profile of coarse conjectures π is *first-order consistent at* $(\bar{\mathbf{p}}, \bar{\alpha})$, where $\bar{\mathbf{p}}$ is an aggregate state and $\bar{\alpha}$ is a profile of actions, if for every player i there are open neighbourhoods $O^i(\bar{\mathbf{p}})$ and $O^i(\bar{\alpha}^i)$ and a smooth function $\mathbf{p}^i : O^i(\bar{\alpha}^i) \to O^i(\bar{\mathbf{p}})$ such that,

(i) $\Pi^i(\alpha^i, \delta^{\neg i}) \cap O^i(\bar{\pi}) = \{\mathbf{p}^i(\alpha^i)\}$ for each $a^i \in O^i(\bar{\alpha}^i)$; and
(ii) $\mathbf{p}^i(\bar{\alpha}^i) = \pi^i(\bar{\alpha}^i, \bar{\alpha}^i, \bar{\mathbf{p}})$ and $\partial \mathbf{p}^i(\bar{\alpha}^i) = \partial_{\alpha^i} \pi^i(\bar{\alpha}^i, \bar{\alpha}^i, \bar{\mathbf{p}})$.

The first condition requires that the function be a local selection of states that are induced, and the second that the conjecture be correct, at the equilibrium action and to a first-order approximation, relative to the local selection. If the conjecture is variational, the second condition simply requires that $\partial \mathbf{p}^i(\bar{\alpha}^i) = \delta^i(\bar{\alpha}^i)$. In the same case, if we maintain the first condition but strengthen the second one to require that $\partial \mathbf{p}^i(\alpha^i) = \delta^i(\alpha^i)$ for each $\alpha^i \in O^i(\bar{\alpha}^i)$, we say that profile δ is *locally consistent around* $(\bar{\mathbf{p}}, \bar{\alpha})$.

A *first-order point-consistent equilibrium* consists of a profile of actions, $\bar{\alpha}$, and a profile of coarse conjectures, $\bar{\pi}$, such that:

(i) $\bar{\alpha}^i \in B^i(\bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha}), \bar{\pi}^i)$ for all i; and
(ii) profile $\bar{\pi}$ is first-order consistent at $(\boldsymbol{\pi}(\bar{\alpha}), \bar{\alpha})$.

Here, the first condition requires that each player's action be stable given the equilibrium aggregate state and her conjecture, and the second condition that player's conjecture be correct from the point of view of its level and first derivative at the equilibrium actions.If all conjectures are variational, and one maintains the first condition but strengthens the second one to local consistency around $(\bar{\mathbf{p}}, \bar{\alpha})$, the profiles of actions and conjectures are similar to the concept of equilibrium defined by Bresnahan (1981).[31]

[31] This definition of equilibrium is slightly more general than the concept introduced in Bresnahan (1981), in the sense that we have not assumed that the aggregate states induced by players are always uniquely determined. Suppose, as he does, that $\Pi^i(\alpha^i, \delta^{\neg i})$ is always a singleton, interpret it as a function, and suppose that this function is smooth; then, the definition of local consistency simply requires that $\partial_{\alpha^i} \Pi^i(\alpha^i, \bar{\delta}^{\neg i}) = \bar{\delta}^i(\alpha^i)$, for all α^i in some open neighborhood of $\bar{\alpha}^i$, for each i.

7.2 Market games and equilibrium

Recall now the financial economy used before in the paper, which evolves over two periods, has an uncertain state of the world, $s = 1, \dots, S$, in the second period, and where in the first period investors trade assets that deliver their return in the second period. Suppose, for the sake of notational simplicity, that the market offers the complete set of elementary securities for trade, and consider the following market game. Let

$$\mathbf{A}^i := \{(b^i, q^i) \in \mathbf{R}_+^{2S} : q^i \le e^i \quad \text{and } b^i \cdot q^i = 0\}, \tag{41}$$

where b_s^i denotes the amount of numèraire that individual i spends on asset s, while q_s^i is the number of units of asset s that she sells. As in the standard market game, define the payoff functions $v^i : \mathbf{A} \to \mathbf{R}$ by

$$U^i(b, q) := -\sum_s b_s^i + \sum_s q_s^i \frac{\sum_{j=1}^J b_s^j}{\sum_{j=1}^J q_s^j} + u^i\left(\left(e_s^i + b_s^i \frac{\sum_{j=1}^J q_s^j}{\sum_{j=1}^J b_s^j} - q_s^i\right)_{s=1}^S\right), \tag{42}$$

where we are dismissing profiles of actions for which $\sum q_s^i = 0$ of $\sum b_s^i = 0$ for some s.[32]

This game can be written in the following aggregative form. Let the aggregator function $\boldsymbol{\pi} : \mathbf{A} \to \mathbf{R}_+^S$ be defined by $\boldsymbol{\pi}_s(b, q) = (\sum_i b_s^i)/(\sum_i q_s^i)$, and let the functions $v^i : \mathbf{A}^i \times \mathbf{R}_+^S \to \mathbf{R}$ be

$$v^i((b^i, q^i), \mathbf{p}) := -\sum_s b_s^i + \sum_s q_s^i \mathbf{p}_s + u^i\left(\left(e_s^i + \frac{b_s^i}{\mathbf{p}_s} - q_s^i\right)_{s=1}^S\right). \tag{43}$$

Now, all the concepts of equilibrium in games presented above can be applied here, relative to the aggregator function $\boldsymbol{\pi}$.

7.2.1 Equilibrium in thin markets and market games

Given a profile $\bar{m}$ of diagonal, positive definite price impact matrices, define, for each individual and asset, the function $\varphi_s^i : \mathbf{R}_+ \times \mathbf{R}_{++} \times \mathbf{R} \to \mathbf{R}$ by

$$\varphi_s^i(b_s, p_s, c_s) := \frac{1}{2}\left(p_s + \bar{m}_s^i(e_s^i - c_s) + \sqrt{(p_s + \bar{m}_s^i(e_s^i - c_s))^2 + 4\bar{m}_s^i b_s}\right). \tag{44}$$

[32] We will maintain this simplification throughout the analysis.

Define also the following coarse conjectures: for individual i,

$$\bar{\pi}^i_s((b^i, q^i), (\bar{b}^i, \bar{q}^i), \bar{p}) := \begin{cases} \bar{p}_s - \bar{m}^i_s(q^i_s - \bar{q}^i_s), & \text{if } \bar{q}^i_s > 0 \text{ and } b^i_s = 0; \\ \varphi^i_s(b_s, \bar{p}_s, e^i_s + \frac{\bar{b}^i_s}{\bar{p}_s} - \bar{q}^i_s) - m^i_s q^i_s, & \text{otherwise.} \end{cases} \tag{45}$$

Henceforth, all computations assume that $\bar{p}_s - \bar{m}^i_s(\bar{c}^i_s - e^i_s) > 0$. Under this assumption $\varphi^i_s(0, p_s, c_s) > 0$, whereas, by direct computation,

$$\varphi^i_s(\bar{p}_s(\bar{c}_s - e^i_s), \bar{p}_s, \bar{c}_s) = \bar{p}_s \tag{46}$$

and

$$\frac{\partial \varphi^i_s}{\partial b_s}(\bar{p}_s(\bar{c}_s - e^i_s), \bar{p}_s, \bar{c}_s) = \frac{\bar{m}^i_s}{\bar{m}^i_s(\bar{c}^i_s - e^i_s) + \bar{p}_s}. \tag{47}$$

Also,[33]

$$\varphi^i_s(b_s, \bar{p}_s, \bar{c}_s) = \bar{p}_s + \bar{m}^i_s\left(\frac{b_s}{\varphi^i_s(b_s, \bar{p}_s, \bar{c}_s)} + e^i_s - \bar{c}^i_s\right). \tag{49}$$

Lemma 7 *For every individual i, and every asset s, equality*

$$\bar{\pi}^i_s((b^i, q^i), (\bar{b}^i, \bar{q}^i), \bar{p}) = \bar{p}_s + \bar{m}^i_s(c^i_s - \bar{c}^i_s) \tag{50}$$

holds whenever $\bar{q}^i_s = (\bar{c}^i_s - e^i_s)_-$, $\bar{b}^i_s = \bar{p}_s(\bar{c}^i_s - e^i_s)_+$, $q^i_s = (c^i_s - e^i_s)_-$ and

$$b^i_s = \bar{\pi}^i_s((b^i, q^i), (\bar{b}^i, \bar{q}^i), \bar{p})(c^i_s - e^i_s)_+. \tag{51}$$

Proof See Appendix A1. □

The following result establishes the relation between our definition of equilibrium in thin financial markets and equilibrium in market games.

Proposition 6 *An equilibrium in thin markets is a first-order point-consistent equilibrium in the market game. That is: let $(\bar{p}, \bar{\theta}, \bar{m})$ be an equilibrium in thin markets of the financial economy, let the profile of actions for the market game be $\bar{b}^i_s = \bar{p}_s(\bar{\theta}^i_s)_+$ and $\bar{q}^i = \bar{\theta}^i_-$, and let the profile of coarse conjectures $\bar{\pi}$ be defined as in Eq. 45; then, $((\bar{b}, \bar{q}), \bar{\pi})$ is a first-order point-consistent equilibrium in the market game.*

The proof of this proposition will use the following lemmas.

[33] To see this, simply notice that $\varphi^i_s(b_s, \bar{p}_s, \bar{c}_s)$ solves, in $\boldsymbol{\pi}$, the quadratic equation

$$\boldsymbol{\pi}^2 - (p_s + \bar{m}^i_s(e^i_s - c^i_s))\boldsymbol{\pi} - \bar{m}^i_s b_s = 0. \tag{48}$$

Lemma 8 *For individual i, fix a diagonal price impact matrix $\bar{m}^i$ and let conjecture $\bar{\pi}^i$ be defined by Eq. 45. Then,*

1. *if trade θ^i is stable for i given $(p, \bar{m}^i)$ in the financial market, then the action defined by $(p_s(\theta^i_s)_+, (\theta^i_s)_-)_{s=1}^S$ is stable for her, given $(p, \bar{\pi}^i)$, in the market game; and*
2. *if action (b^i, q^i) is stable for i given $(p, \bar{\pi}^i)$ in the market game, then the trade defined by $(\frac{b^i_s}{p_s} - q^i_s)_{s=1}^S$ is stable for her, given $(p, \bar{m}^i)$, in the financial market.*

Proof See Appendix A1. □

Lemma 9 *For a given individual i, suppose that all price impact matrices $\bar{m}^{\neg i}$ are diagonal, and let the conjectures $\bar{\pi}^{\neg i}$ be defined as in Eq. 45. Then,*

1. *if $p \in \mathcal{P}^i(\theta^i; \bar{m}^{\neg i})$, then*

$$p \in \Pi^i((p_s(\theta^i_s)_+, (\theta^i_s)_-)_{s=1}^S, \bar{\pi}^{\neg i}); \tag{52}$$

2. *if $p \in \Pi^i((b^i, q^i), \bar{\pi}^{\neg i})$, then*

$$p \in \mathcal{P}^i((e^i_s + \frac{b^i_s}{p_s} - q^i_s)_{s=1}^S, \bar{m}^{\neg i}). \tag{53}$$

Proof See Appendix A1. □

Proof of Proposition 6 By Lemma 8 and Eq. 46, we only need to prove that profile $\bar{\pi}$ is first-order consistent at $(\bar{p}, (\bar{b}, \bar{q}))$. Fix any individual i. If $\bar{q}^i_s > 0$, the argument for asset s is immediate. Now consider an asset for which $\bar{b}^i_s > 0$. Let P^i be the local inverse demand around $\bar{\theta}^i$. Since $\bar{m}^i$ is positive definite and diagonal, in a neighborhood of $\bar{c}^i_s = e^i_s + \bar{\theta}^i_s$, function $\beta_s(c^i) := p^i_s(c^i_s - e^i_s)(c^i_s - e^i_s)$ is a bijection into a neighborhood of $\bar{b}^i_s$. Define $\mathbf{p}^i_s(b^i_s) = P^i_s(\beta^{-1}(b^i_s))$, over that neighborhood. That $\mathbf{p}^i$ is a selection of Π^i and is locally unique follows from Lemma 9. Now, by construction, $\mathbf{p}^i_s(\bar{b}^i_s) = \bar{p}_s$, while, by direct computation,

$$\partial \mathbf{p}^i_s(\bar{b}^i_s) = \frac{\partial P^i_s(\bar{c}^i_s - e^i_s)}{\partial \beta_s(\bar{c}^i_s)} = \frac{\bar{m}^i_s}{\bar{m}^i_s(\bar{c}^i_s - e^i_s) + \bar{p}_s}, \tag{54}$$

so the result follows from Eqs. 46 and 47. □

7.2.2 Nash equilibrium in market games and financial equilibrium

We now consider the opposite exercise: we determine what kind of conjectures the traders would have to have in the financial markets, if they traded as in the Nash equilibrium of the market game. To do this, define, for each individual i, given a vector of prices $\bar{p}$ and a profile of trades for all other individuals $\bar{\theta}^{\neg i}$, the function

$$P^i(\cdot; \bar{p}, \bar{\theta}^{\neg i}) : \left\{ \theta^i \in \mathbf{R}^S_+ : \theta^i \ll \sum_{j \neq i} (\bar{\theta}^j)_- \right\} \to \mathbf{R}^S \tag{55}$$

as follows: for each asset s,

$$P^i_s(\theta^i; \bar{p}, \bar{\theta}^{\neg i}) := \frac{\bar{p}_s \sum_{j \neq i} (\bar{\theta}^j_s)_+}{(\theta^i_s)_- + \sum_{j \neq i} (\bar{\theta}^j_s)_-}. \tag{56}$$

That is, individual i finds prices p for which there exist trades for all other individuals, $\theta^{\neg i}$, such that[34]

(i) for all $j \neq i$ and all s, $p_s(\theta^j_s)_+ = \bar{p}_s(\bar{\theta}^j_s)_+$;
(ii) for all $j \neq i$ and all s, $(\theta^j_s)_- = (\bar{\theta}^j_s)_-$; and
(iii) for all s, $\sum_{j \neq i}(\theta^j_s) = -\theta^i_s$.

Now, say that an *accounting-thin equilibrium* of the financial market is a vector of prices $\bar{p} \in \mathbf{R}^S$ and a profile of asset trades, $\bar{\theta} \in \mathbf{R}^{LI}_+$, such that

(i) trade $\bar{\theta}^i$ solves the problem

$$\max_{\theta^i} -P^i(\theta^i, \bar{p}, \bar{\theta}^{\neg i}) \cdot \theta^i + u^i(e^i + \theta^i) \tag{58}$$

for all i; and
(ii) $\sum_i \theta^i = 0$.

Here, the first condition requires that all individuals consider their trades optimal, given the effect on prices that the accounting function predicts. The next result establishes the relation between this definition of equilibrium and the concepts of equilibrium in market games.

Proposition 7 *Accounting-thin equilibrium in markets and Nash equilibrium in the market game are equivalent. That is:*

1. *if $(\bar{p}, \bar{\theta})$ is an accounting-thin equilibrium of the financial market, then its associated profile of actions $(\bar{b}, \bar{q})$ is a Nash equilibrium of the market game; and*
2. *if $(\bar{b}, \bar{q})$ is a Nash equilibrium of the market game, then its associated pair of prices and consumptions $(\bar{p}, \bar{\theta})$ is an accounting-thin equilibrium of the financial market.*

Proof Let $\bar{\delta}$ be the profile of variational conjectures defined by

$$\bar{\delta}^i(b^i, q^i) := \partial_{(b^i, q^i)} \boldsymbol{\pi}((b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i})), \tag{59}$$

[34] Intuitively, individual i does market clearing and finds prices subject to the condition that the nominal value of the consumption of each net buyer and the consumption of each net seller of asset s remain unchanged: Prices $P^i(\theta^i; \bar{p}, \bar{\theta}^{\neg i})$ solve the system along with trades defined as follows: for all $j \neq i$ and all s,

$$\theta^j_s = \begin{cases} \bar{\theta}^j_s, & \text{if } \theta^j_s \leq 0; \\ \frac{\bar{p}_s \bar{\theta}^j_s}{P^i_s(\theta^i; \bar{p}, \bar{\theta}^{\neg i})}, & \text{otherwise.} \end{cases} \tag{57}$$

and notice that, under these conjectures,

$$B^i((\bar{b}^i, \bar{q}^i), \boldsymbol{\pi}(\bar{b}, \bar{q}), \bar{\delta}^i) = \text{argmax}_{(b^i,q^i)} v^i[(b^i, q^i), \boldsymbol{\pi}((b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i}))], \quad (60)$$

for every individual i.

Now, for the first part, by Lemma 6 it suffices to show that each $(\bar{b}^i, \bar{q}^i)$ is in the set defined in (60). If this is not the case, one can find an action (b^i, q^i) such that

$$v^i((b^i, q^i), \boldsymbol{\pi}((b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i})) > v^i((\bar{b}^i, \bar{q}^i), \boldsymbol{\pi}(\bar{b}, \bar{q})). \quad (61)$$

For simplicity of notation, let $\bar{\pi} = \boldsymbol{\pi}(\bar{b}, \bar{q})$ and $\pi = \boldsymbol{\pi}((b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i}))$. Define θ^i by $\theta^i_s = b^i_s/\pi_s - q^i_s$, and notice that, by construction,[35] $P^i(\theta^i, \bar{p}, \bar{\theta}^{\neg i}) = \pi$ and $P^i(\bar{\theta}^i, \bar{p}, \bar{\theta}^{\neg i}) = \bar{\pi} = \bar{p}$. By direct substitution, we then get that

$$\begin{aligned} -P^i(\theta^i; \bar{p}, \bar{\theta}^{\neg i}) \cdot \theta^i + u^i(e^i + \theta^i) &= v^i((b^i, q^i), \pi) \\ &> v^i((\bar{b}^i, \bar{q}^i), \bar{\pi}) \\ &= -P^i(\bar{\theta}^i; \bar{p}, \bar{\theta}^{\neg i}) \cdot \bar{\theta}^i + u^i(e^i + \bar{\theta}^i), \end{aligned}$$

which is impossible.

For the second part, suppose that it is not true that $\bar{\theta}^i$ solves the problem

$$\max_{\theta^i} -P^i(\theta^i, \bar{p}, \bar{\theta}^{\neg i}) \cdot \theta^i + u^i(e^i + \theta^i), \quad (65)$$

and fix a θ^i such that

$$-P^i(\theta^i, \bar{p}, \bar{\theta}^{\neg i}) \cdot \theta^i + u^i(e^i + \theta^i) > -P^i(\bar{\theta}^i, \bar{p}, \bar{c}^{\neg i}) \cdot \bar{\theta}^i + u^i(e^i + \bar{\theta}^i). \quad (66)$$

[35] To see this, notice first that if $b^i_s = 0$, then

$$P^i_s(\theta^i; \bar{p}, \bar{\theta}^{\neg i}) = \frac{\sum_{j \neq i} \bar{b}^j_s}{q^i_s + \sum_{j \neq i} \bar{q}^j_s} = \pi_s. \quad (62)$$

Also, notice that if $b^i_s > 0$, then

$$\pi_s = \frac{b^i_s + \sum_{j \neq i} \bar{b}^j_s}{\sum_{j \neq i} \bar{q}^j_s} = \frac{\pi_s \theta^i_s + \bar{p}_s \sum_{j \neq i} (\bar{\theta}^j_s)_+}{\sum_{j \neq i} (\bar{\theta}^j_s)_-}, \quad (63)$$

which implies that

$$\pi_s = \frac{\bar{p}_s \sum_{j \neq i} (\bar{\theta}^j_s)_+}{\sum_{j \neq i} (\bar{\theta}^j_s)_+ - \theta^i_s} = P^i_s(c^i; \bar{p}, \bar{c}^{\neg i}). \quad (64)$$

The other equality is by direct computation.

Let $p = P^i(\theta^i, \bar{p}, \bar{\theta}^{\neg i})$ and recall that $P^i(\bar{\theta}^i, \bar{p}, \bar{\theta}^{\neg i}) = \bar{p}$. Define $b^i_s = p_s(\theta^i_s)_+$ and $q^i_s = (\theta^i_s)_-$, and notice that, again by construction,[36] $\boldsymbol{\pi}((b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i})) = p$ and $\boldsymbol{\pi}(\bar{b}, \bar{q}) = \bar{p}$. It follows then that

$$\begin{aligned} v^i[(b^i, q^i), \boldsymbol{\pi}(b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i})] &= -p \cdot \theta^i + u^i(e^i + \theta^i) \\ &> -\bar{p} \cdot \bar{\theta}^i + u^i(e^i + \bar{\theta}^i) \\ &= v^i((\bar{b}^i, \bar{q}^i), \boldsymbol{\pi}(\bar{b}, \bar{q})), \end{aligned}$$

again an impossibility. □

7.3 No arbitrage, thin markets and market games

The results obtained in this paper contrast with the message given by Koutsougeras (2003) and Koutsougeras and Papadapoulos (2004). There, an example is provided of a non-competitive setting where arbitrage opportunities may exist at equilibrium: at the Nash equilibrium of a strategic market game, asset prices (would) provide an arbitrage opportunity.[37] The analysis above explains the conceptual differences between the two concepts of equilibrium, and therefore the quantitative differences in their predictions, but does not clarify a key distinction. Note that in a market game, a deviation of an agent in a given market only affects that market's price. At a Nash equilibrium (i.e. in the accounting-thin equilibrium above), individuals thus have diagonal price impact matrices, whatever the asset structure of the economy. In our concept of equilibrium, traders would have diagonal matrices when the only traded assets are the full set of

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[36] To see this, notice that if $\theta^i_s \leq 0$, then

$$\boldsymbol{\pi}_s((b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i})) = \frac{\sum_{j \neq i} \bar{p}_s(\bar{\theta}^i_s)_+}{\sum_{j \neq i}(\bar{\theta}^i_s)_- - \theta^i_s} = p_s, \tag{67}$$

while if $\theta^i_s > 0$ one has that

$$p_s = \frac{\sum_{j \neq i} b^j_s}{-\frac{b^i_s}{p_s} + \sum_{j \neq i} q^j_s}, \tag{68}$$

which implies that

$$p_s = \frac{b^i_s + \sum_{j \neq i} b^j_s}{\sum_{j \neq i} q^j_s} = \boldsymbol{\pi}_s((b^i, q^i), (\bar{b}^{\neg i}, \bar{q}^{\neg i})). \tag{69}$$

The other inequality is immediate.

[37] The example below does not give us this result. To see this, suppose that the economy is endowed with the complete set of elementary securities. Then, at the only symmetric Nash equilibrium there is no trade of the security for state 3, while the first two are traded at an equilibrium price of 5/12. On the other hand, if the traded assets are as in the example of the previous section, the risk free asset is not traded while the other two securities are traded at a price of 1.43. None of these results exhibits an arbitrage opportunity, even if we compare them: the implicit price for state 3 is $(1.43 - 2(5/6))/2 > 0$.

elementary securities, but not for general asset structures (see part 1 of Lemma 5). This means that, in general, the price impacts implicit in the Nash equilibria of a market game do give irrelevant impact, while we have shown the existence of equilibria in thin markets under the requirements that all price impact matrices give no irrelevant impact. We conjecture, but leave for future research, that all models of non-competitive trading where the implicit price impact matrices give no irrelevant impact rule out, at least, weak arbitrage opportunities, so that our concept of equilibrium lies in this class, but the concept of Nash equilibrium in market games does not.

8 Concluding remarks

We have studied a model of trade in financial markets where individual investors recognize the fact that prices do depend on their trades. In this setting, the argument why asset prices do not allow for arbitrage opportunities at equilibrium fails. In the customary argument, arbitrage opportunities cannot exist because if they do, the portfolios that individuals are demanding cannot be optimal: whatever her preferences and beliefs[38] adding one unit of the arbitrage opportunity to the portfolio of one trader would make her strictly better-off. Thus, according to this argument, equilibrium asset prices eliminate arbitrage opportunities and embed, in consequence, an objective probability distribution that allows for the pricing of any state-contingent claim in the economy to be computed as the discounted expected return it entails. But, indeed, this argument is untenable if individual investors do anticipate that their trades will affect asset prices: even if an arbitrage opportunity exists, it may be that an investor's attempt to exploit it affects the prices in a way such that the cost of her existing portfolio increases to the point that she is no longer better-off in ex-ante terms. If a theory of asset pricing based on an objective probability embedded in prices is going to be consistent with noncompetitive behavior of investors, a new argument has to be provided.

In this paper, we have tackled that question in the context of a two-period, financial economy with a finite number of states of the world. We have considered a situation in which all traders know that, given a *status quo* of prices and trades in the market, if they were to attempt a different trade, they would (have to) affect prices in order to guarantee that the rest of the market is willing to accommodate their increased demands or sales. In such a case, the predictions of models where all traders follow price-taking behavior are inapplicable. For instance, each individual's willingness to trade is lower than in the competitive case, and, consequently, not all gains to trade are exhausted and trade leads to an equilibrium in which the asset allocation is Pareto-inefficient.[39] Here, we have shown that, even in noncompetitive financial markets, financial equilibria that preclude arbitrage opportunities do exist.

Some assumptions were made. First, we only consider agents with preferences that have von Neumann–Morgenstern representation with respect to future consumption,

[38] Assuming, of course, monotonicity of ex-ante preferences on consumption in all future states of the world.

[39] As long as a priori, there are some gains to trade, i.e. the initial endowments are Pareto inefficient. The deadweight loss due to non-competitive trading depends negatively on the depth of the market, and it vanishes completely as a number of traders approaches infinity (see Weretka 2007a).

and that are quasilinear with respect to date-zero consumption, a variable in which we impose no non-negativity constraints; effectively, these assumptions leave all future consumption free of income effects, and simplifies substitution effects across consumption in different states of the world, which makes our mathematical problem more tractable. Secondly, we consider only the case in which the existing financial markets allow for complete insurance opportunities against risk; this allows us to obtain an auxiliary representation of the economy by replacing the financial markets with a complete set of elementary securities, in which, thanks to the separability of preferences, we can restrict attention to the case when price cross-effects are null: each agent beliefs, a posteriori correctly, that if she expands her order for some security, only the price of that security will be affected.

We then invoke the argument of Weretka (2007a) to prove the existence of equilibria in the auxiliary economy in which no arbitrage opportunities can exist. This conclusion, though, follows as a consequence of the facts that all traders are individually rational and all markets are clearing, which contrasts with the competitive case, in which the existence of just one individually rational trader suffices, regardless of market clearing. We then associate the equilibria of the auxiliary economy with equilibria on the original financial structure, and obtain that at these equilibria no arbitrage opportunities can exist either. Furthermore, we provide an extension of the fundamental theorem of asset pricing to (two-period, finite) noncompetitive economies, which allow not only for the computation of the prices of redundant securities, but also for the direct determination of the price impacts exerted by these securities (and also of those exerted by other securities on these). The effect of market thinness on asset prices is not clear without further assumptions. If one assumes a symmetric economy with CRRA preferences, state prices are above the ones that would prevail were the markets competitive.[40]

Critically, in our analysis market power is determined endogenously, as part of the definition of equilibrium. Here, we have considered the case in which traders correctly estimate their price impacts to a first-order level of accuracy: implicit in their individual portfolio problems, they impute a correct linear approximation to the real inverse demand they face from the rest of the market (which, in itself, depends on the estimations other traders are making of the inverse demand they face, estimations that depend themselves on the one made by the trader in question). This assumption simplifies the definition and treatment of equilibrium, in that it guarantees that each trader's solution to her portfolio problem is characterized by its first-order conditions, but is of no further importance. Moreover, Weretka (2007b) provides strategic foundations for the equilibrium used here.

The relaxation of the assumptions made here and verification of the validity of non-competitive mechanisms in generating the asset pricing puzzles remain topics for further research. If one guarantees that the second-order conditions of the portfolio problem are satisfied for each trader, then the assumption that they only estimate their inverse demands to a first-order level of accuracy can be removed. The assumptions of separability and quasilinearity are useful, but ought to be relaxed.

[40] For the detailed study of model predictions for different market structures and utility functions, see Weretka (2007a).

Appendix A1: Lemmata

Proof of Lemma 1 For necessity, the first two conditions are immediate, and we only need to concentrate on the third condition.

For any investor j, since Θ^j is stable given $(\bar{P}, \bar{M}^j)$ it must be that it solves the equation $\bar{P} = R^\top \partial u^j(e^j + R\bar{\Theta}^j) - \bar{M}^j\bar{\Theta}^j$. By strong concavity of preferences, non-redundancy of assets and positive definiteness of the price impact matrix, this defines, locally around $\bar{P}$, a differentiable individual (stable) demand function $\Theta^j(P; \bar{M}^j)$, with derivative

$$\partial_P \Theta^j(\bar{P}; \bar{M}^j) = (R^\top \partial^2 u^j(e^j + R\bar{\Theta}^j)R - \bar{M}^j)^{-1}, \tag{70}$$

a negative definite matrix.

Thus, the demand trader i faces from the rest of the market is, locally around $\bar{P}$, $\varphi^i(P) = \sum_{j \neq i} \Theta^j(P; \bar{M}^j)$. By construction,

$$\partial\varphi^i(\bar{P}) = \sum_{j \neq i} (R^\top \partial^2 u^j(e^j + R\bar{\Theta}^j)R - \bar{M}^j)^{-1}, \tag{71}$$

again a negative definite matrix. A subequilibrium triggered by a local deviation $\tilde{\Theta}^i$ (close enough to $\bar{\Theta}^i$) requires prices $\tilde{P}$ such that $\varphi^i(\tilde{P}) = -\tilde{\Theta}^i$. This defines, locally, a differentiable subequilibrium price function $\phi^i(\tilde{\Theta}^i)$, with the property that

$$\partial\phi^i(\bar{\Theta}^i) = -(\partial\varphi^i(\bar{P}))^{-1} = \frac{1}{I-1}\mathcal{H}((\bar{M}^j - R^\top \partial^2 u^j(e^j + R\bar{\Theta}^j)R)_{j \neq i}). \tag{72}$$

Now, since $\bar{M}$ is mutually consistent given $(\bar{P}, \bar{\Theta})$, there exists a local inverse demand function P^i. Since subequilibria are locally unique, it must be that, in a neighborhood of $\bar{\Theta}^i$, $\phi^i(\tilde{\Theta}^i) = P^i(\tilde{\Theta}^i)$, which immediately implies, again by mutual consistency of M, that

$$M^i = \partial P^i(\bar{\Theta}^i) = \partial\phi^i(\bar{\Theta}^i) = \frac{1}{I-1}\mathcal{H}((M^j - R^\top \partial^2 u^j(e^j + R\bar{\Theta}^j)R)_{j \neq i}). \tag{73}$$

For sufficiency, market clearing and stability of all trades are immediate, and we only need to prove mutual consistency of profile $\bar{M}$.

As in the proof of necessity, by the second condition of the lemma, for each trader i we again have, locally around $\bar{P}$, an inverse demand function $\varphi^i(P) = \sum_{j \neq i} \Theta^j(P; \bar{M}^j)$, with

$$\partial\varphi^i(\bar{P}) = \sum_{j \neq i} (R^\top \partial^2 u^j(e^j + R\bar{\Theta}^j)R - \bar{M}^j)^{-1} \tag{74}$$

a negative definite matrix. By the implicit function theorem we can find neighborhoods $N^i(\bar{\Theta}^i)$ and $N^i(\bar{P})$, and a diffeomorphism $\rho^i : N^i(\bar{\Theta}^i) \to N^i(\bar{P})$, such that

(a) $P^i(\tilde{\Theta}^i)$ is the only P in $N^i(\bar{P})$ satisfying that $\varphi^i(P) + \tilde{\Theta}^i = 0$; and
(b) $\partial P^i(\bar{\Theta}^i) = -(\sum_{j \neq i}(R^\top \partial^2 u^j(e^j + R\bar{\Theta}^j)R - \bar{M}^j)^{-1})^{-1}$.

By property (a), for every $\tilde{\Theta}^i \in N(\bar{\Theta}^i)$, $\mathcal{P}^i(\tilde{\Theta}^i; M^{-i}) \cap N(\bar{P}) = \{P^i(\tilde{\theta}^i)\}$, while, by property (b) and the third condition of the lemma, $\partial P^i(\bar{\Theta}^i) = M^i$. □

Proof of Lemma 2 That prices in the set allow no weak arbitrage opportunities is straightforward. Now, vector P allows no weak arbitrage opportunities if, and only if, there exists no Θ for which $P \cdot \Theta < 0$ and $R\Theta = 0$. It follows from Farkas's Lemma that if P allows no weak arbitrage opportunities, then for some $(p_0, p_1) \in \mathbf{R}_{++} \times \mathbf{R}^S$, it is true that $p_0 P = R^\top p_1$. Letting $p = \frac{1}{p_0} p_1$ completes the proof. □

Proof of Lemma 3 Write

$$M = \begin{pmatrix} M_S & \Delta \\ \Delta^\top & \Gamma \end{pmatrix}. \tag{75}$$

Define

$$T = \begin{pmatrix} -R_S^{-1} R_{A-S} \\ \mathbf{I}_{A-S} \end{pmatrix}, \tag{76}$$

and notice that $RT = 0$. Since M gives no irrelevant impact, it follows that $MT = 0$, which implies that $M_S R_S^{-1} R_{A-S} = \Delta$ and $\Delta^\top R_S^{-1} R_{A-S} = \Gamma$.

Now, by direct computation,

$$R^\top m R = \begin{pmatrix} M_S & R_S^\top m R_{A-S} \\ R_{A-S}^\top m R_S & R_{A-S}^\top m R_{A-S} \end{pmatrix}, \tag{77}$$

whereas $R^\top m R T = 0$, so $M_S R_S^{-1} R_{A-S} = R_S^\top m R_{A-S}$. It follows that $R_S^\top m R_{A-S} = \Delta$, which implies that $R_{A-S}^\top m R_{A-S} = \Delta^\top R_S^{-1} R_{A-S} = \Gamma$, and hence that $R^\top m R = M$.

For uniqueness, suppose that $R^\top \tilde{m} R = M$. Then,

$$\tilde{m} = (R_S^\top)^{-1} R_S^\top \tilde{m} R_S (R_S)^{-1} = (R_S^\top)^{-1} M_S (R_S)^{-1} = m. \tag{78}$$

□

Proof of Lemma 4 By construction,

$$\sum_{j \neq i} (\tilde{\Theta}^j + (I-1)^{-1}(\tilde{\Theta}^i - \hat{\Theta}^i)) + \hat{\Theta}^i = \sum_{j \neq i} \tilde{\Theta}^j + \tilde{\Theta}^i - \hat{\Theta}^i + \hat{\Theta}^i = 0, \tag{79}$$

and hence all markets clear. It only remains to show that for each $j \neq i$, trade $\tilde{\Theta}^j + (I-1)^{-1}(\tilde{\Theta}^i - \hat{\Theta}^i)$ is stable for j given $(\tilde{P}, \bar{M}^j)$. Suppose not: for some j and some Θ^j,

$$-(\tilde{P} + \bar{M}^j(\Theta^j - (\tilde{\Theta}^j + (I-1)^{-1}(\tilde{\Theta}^i - \hat{\Theta}^i)))) \cdot \Theta^j + u^j(e^j + R\Theta^j)$$
$$> -\tilde{P} \cdot (\tilde{\Theta}^j + (I-1)^{-1}(\tilde{\Theta}^i - \hat{\Theta}^i)) + u^j(e^j + R(\tilde{\Theta}^j + (I-1)^{-1}(\tilde{\Theta}^i - \hat{\Theta}^i))).$$

Since $R(\tilde{\Theta}^i - \hat{\Theta}^i) = 0$ and $\bar{M}^j$ gives no irrelevant price impact, it follows that $\bar{M}^j(\tilde{\Theta}^i - \hat{\Theta}^i) = 0$ and, by Proposition 1, $\tilde{P} \cdot (\tilde{\Theta}^i - \hat{\Theta}^i) = 0$. Then, the previous equation is equivalent to

$$-(\tilde{P} + \bar{M}^j(\Theta^j - \tilde{\Theta}^j)) \cdot \Theta^j + u^j(e^j + R\Theta^j) > -\tilde{P} \cdot \tilde{\Theta}^j + u^j(e^j + R\tilde{\Theta}^j), \tag{80}$$

which means that $\tilde{\Theta}^j$ is not stable for j given $(\tilde{P}, \bar{M}^j)$, contradicting the fact that $(\tilde{P}, \tilde{\Theta}^{-i})$ is a subequilibrium triggered by trade $\tilde{\Theta}^i$. □

Proof of Lemma 5 For clarity, we denote by $\mathcal{S}^i(\Theta^i; M^{-i}, R)$ and $\mathcal{P}^i(\Theta^i; M^{-i}, R)$ the sets of subequilibria triggered by Θ^i, given M^{-i}, and its projection into the space of prices, when the economy is endowed with asset structure R. We distinguish these sets for the case of elementary securities, by denoting them as $\mathcal{S}^i(\Theta^i; M^{-i}, \mathbf{I}_S)$ and $\mathcal{P}^i(\Theta^i; M^{-i}, \mathbf{I}_S)$.

Notice first that, since each $(\bar{M}^i, \bar{m}^i)$ is R-associated, it is immediate that all $\bar{M}^i$ give no irrelevant price impacts. Also, notice that, by Lemma 3, $\bar{m}^i = (R_S^{-1})^\top \bar{M}_S^i R_S^{-1}$ for all i.

For the first claim, suppose that $(\bar{p}, \bar{\theta}, \bar{m})$ is an equilibrium for $\{u, e, \mathbf{I}_S\}$ and $\sum_i \bar{\Theta}^i = 0$. Market clearing is assumed, so we only need to show that each $\bar{\Theta}^i$ is stable for i given $(\bar{P}, \bar{M}^i)$, and that the profile $\bar{M}$ is mutually consistent.

Suppose that $\bar{\Theta}^i$ is not stable for i given $(\bar{P}, \bar{M}^i)$. Then, we can fix Θ such that

$$-(\bar{P} + \bar{M}^i(\Theta - \bar{\Theta}^i)) \cdot \Theta + u^i(e^i + R\Theta) > -\bar{P} \cdot \bar{\Theta}^i + u^i(e^i + R\bar{\Theta}^i). \tag{81}$$

Given that $(\bar{P}, \bar{\Theta}, \bar{M})$ and $(\bar{p}, \bar{\theta}, \bar{m})$ are R-associated, it follows that

$$-(R^\top \bar{p} + R^\top \bar{m}^i R(\Theta - \bar{\Theta}^i)) \cdot \Theta + u^i(e^i + R\Theta) >$$
$$-(R^\top \bar{p}) \cdot \bar{\Theta}^i + u^i(e^i + R\bar{\Theta}^i), \tag{82}$$

so, letting $\theta = R\Theta$, by direct computation,

$$-(\bar{p} + \bar{m}^i(\theta - \bar{\theta}^i)) \cdot \theta + u^i(e^i + \theta) > -\bar{p} \cdot \bar{\theta}^i + u^i(e^i + \bar{\theta}^i), \tag{83}$$

contradicting the fact that, in the economy with elementary securities, $\bar{\theta}^i$ is stable for i given $(\bar{p}, \bar{m}^i)$.

Now, for each i, by mutual consistency of profile $\bar{m}$, there exist neighborhoods $N^i(\bar{\theta}^i)$ and $N^i(\bar{p})$ and a differentiable, local inverse demand function $p^i : N^i(\bar{\theta}^i) \rightarrow$

$N^i(\bar{p})$ such that $p^i(\bar{\theta}^i) = \bar{p}$ and $\partial p^i(\bar{\theta}^i) = \bar{m}^i$. Then, we can define open neighborhoods

$$N^i(\bar{\Theta}^i) = \{\Theta | R\Theta \in N^i(\bar{\theta}^i)\} \tag{84}$$

and

$$N^i(\bar{P}) = \{P | (R_S^\top)^{-1}(P_1, \ldots, P_S)^\top \in N^i(\bar{p})\}. \tag{85}$$

By construction, $f(\Theta) = R\Theta$ maps $N^i(\bar{\Theta}^i)$ into $N^i(\bar{\theta}^i)$, and $g(p) = R^\top p$ maps $N^i(\bar{p})$ into $N^i(\bar{P})$, so we can define $P^i = g \circ p^i \circ f : N^i(\bar{\Theta}^i) \to N^i(\bar{P})$. First, notice that $P^i(\bar{\Theta}^i) = R^\top p^i(R\bar{\Theta}^i) = R^\top \bar{p} = \bar{P}$, while

$$\partial P^i(\bar{\Theta}^i) = R^\top \partial p^i(\bar{\theta}^i) R = R^\top \bar{m}^i R = \bar{M}^i. \tag{86}$$

We now want to show that for any $\Theta \in N^i(\bar{\Theta}^i)$, $P^i(\Theta)$ is the unique (locally) subequilibrium price triggered by $\Theta^i = \Theta$, namely that $\mathcal{P}^i(\Theta; \bar{M}^i, R) \cap N^i(\bar{P}) = \{P^i(\Theta)\}$. First, notice that in the economy with elementary securities, for some θ^{-i}, $(p^i(R\Theta), \theta^{-i}) \in \mathcal{S}^i(R\Theta; \bar{m}^{-i}, \mathbf{I}_S)$. Defining

$$\Theta^j = \begin{pmatrix} R_S^{-1}(\theta^j + \frac{1}{I-1} R\Theta) \\ 0 \end{pmatrix} - \frac{1}{I-1}\Theta, \tag{87}$$

for each $j \neq i$, it follows that $(P^i(\Theta), (\Theta^j)_{j \neq i}) \in \mathcal{S}^i(\Theta; \bar{M}^i, R)$, and hence that $P^i(\Theta) \in \mathcal{P}^i(\Theta^i; \bar{M}^{-i}, R)$. Now, suppose that there exists another $P \in \mathcal{P}^i(\Theta; \bar{M}^{-i}, R) \cap N^i(\bar{P})$. Then, for some $\tilde{\Theta}^{-i}$, $(P, \tilde{\Theta}^{-i}) \in \mathcal{S}^i(\Theta; \bar{M}^{-i}, R)$. Since each $\bar{M}^j$ gives no irrelevant price impact, it follows from Proposition 1 that P allows no weak arbitrage opportunities, and then, from Lemma 2, that there exists $p \in \mathbf{R}^S$ such that $R^\top p = P$. By construction, $p = (R_S^\top)^{-1}(P_1, \ldots, P_S)^\top \in N^i(\bar{p})$ and

$$((R_S^\top)^{-1} P, (R\tilde{\Theta}^j)_{j \neq i}) \in \mathcal{S}^i(R\Theta; \bar{m}^{-i}, \mathbf{I}_S). \tag{88}$$

Also by construction, $p \neq p^i(R\Theta)$ and $p \in N^i(\bar{p}) \cap \mathcal{P}^i(R\Theta; \bar{m}^{-i}, \mathbf{I}_S)$, which is impossible. It follows that for each i, $P^i : N^i(\bar{\Theta}^i) \to N^i(\bar{P})$ is a (differentiable) local inverse demand: for all $\Theta \in N^i(\bar{\Theta}^i)$,

$$\mathcal{P}^i(\Theta; \bar{M}^{-i}, R) \cap N^i(\bar{P}) = \{P^i(\Theta)\}; \tag{89}$$

since $\partial P^i(\bar{\Theta}^i) = \bar{M}^i$, it follows that $\bar{M}$ is mutually consistent.

For the second claim, suppose that $(\bar{P}, \bar{\Theta}, \bar{M})$ is an equilibrium for $\{u, e, R\}$. Market clearing is immediate: $\sum_i \bar{\theta}^i = \sum_i R\bar{\Theta}^i = R \sum_i \bar{\Theta}^i = 0$. As before, we now show that each $\bar{\theta}^i$ is stable for i given $(\bar{p}, \bar{m}^i)$, and that $\bar{m}$ is mutually consistent.

Suppose that $\bar{\theta}^i$ is not stable for i given $(\bar{p}, \bar{m}^i)$. Then, we can fix θ such that

$$-(\bar{p} + \bar{m}^i(\theta - \bar{\theta}^i)) \cdot \theta + u^i(e^i + \theta) > -\bar{p} \cdot \bar{\theta}^i + u^i(e^i + \bar{\theta}^i). \tag{90}$$

Fix any Θ such that $R\Theta = \theta$. Given that $(\bar{P}, \bar{\Theta}, \bar{M})$ and $(\bar{p}, \bar{\theta}, \bar{m})$ are R-associated, it follows by direct computation that

$$-(\bar{P} + \bar{M}^i(\Theta - \bar{\Theta}^i)) \cdot \Theta + u^i(e^i + R\Theta) > -\bar{P} \cdot \bar{\Theta}^i + u^i(e^i + R\bar{\Theta}^i). \quad (91)$$

contradicting the fact that, in the economy with market R, $\bar{\Theta}^i$ is stable for i given $(\bar{P}, \bar{M}^i)$.

Now, for each i, by mutual consistency of profile $\bar{M}$, there exist neighborhoods $N^i(\bar{\Theta}^i)$ and $N^i(\bar{P})$ and a differentiable local subequilibrium price function $P^i : N^i(\bar{\Theta}^i) \to N^i(\bar{P})$ such that $P^i(\bar{\Theta}^i) = \bar{P}$ and $\partial P^i(\bar{\Theta}^i) = \bar{M}^i$. Then, we can define open neighborhoods

$$N^i(\bar{\theta}^i) = \left\{\theta | \begin{pmatrix} R_S^{-1}(\theta - \bar{\theta}^i) \\ 0 \end{pmatrix} + \bar{\Theta}^i \in N^i(\bar{\Theta}^i)\right\} \quad (92)$$

and

$$N^i(\bar{p}) = \{p | R^{\mathsf{T}} p \in N^i(\bar{P})\}. \quad (93)$$

By construction

$$F(\theta) = \begin{pmatrix} R_S^{-1}(\theta - \bar{\theta}^i) \\ 0 \end{pmatrix} + \bar{\Theta}^i \quad (94)$$

maps $N^i(\bar{\theta}^i)$ into $N^i(\bar{\Theta}^i)$, and $G(P) = (R_S^{\mathsf{T}})^{-1}(P_1, \ldots, P_S)^{\mathsf{T}}$ maps $N^i(\bar{P})$ into $N^i(\bar{p})$, so we can define $p^i = G \circ P^i \circ F : N^i(\bar{\theta}^i) \to N^i(\bar{p})$. Notice that $p^i(\bar{\theta}^i) = (R_S^{\mathsf{T}})^{-1} P^i(\bar{\Theta}^i) = \bar{p}$, while

$$\partial p^i(\bar{\theta}^i) = \left((R_S^{-1})^{\mathsf{T}} \; 0\right) \partial P^i(\bar{\Theta}^i) \begin{pmatrix} R_S^{-1} \\ 0 \end{pmatrix} = (R_S^{-1})^{\mathsf{T}} \bar{M}_S^i R_S^{-1} = \bar{m}^i. \quad (95)$$

It only remains to show that for any $\theta \in N^i(\bar{\theta}^i), \mathcal{P}^i(\Theta; \bar{M}^i, \mathbf{I}_S) \cap N^i(\bar{p}) = \{p^i(\theta)\}$. Let $\Theta = F(\theta)$. Notice that in the economy with market R, for some Θ^{-i}, $(P^i(\Theta), \Theta^{-i}) \in \mathcal{S}^i(\Theta; \bar{M}^{-i}, R)$. Defining, for each $j \neq i, \theta^j = R\Theta^j$, it follows that $(p^i(\theta), (\theta^j)_{j \neq i}) \in \mathcal{S}^i(\theta; \bar{m}^i, \mathbf{I}_S)$, and hence that $p^i(\theta) \in \mathcal{P}^i(\theta; \bar{m}^i, \mathbf{I}_S)$. Now, suppose that there exists another $p \in \mathcal{P}^i(\theta; \bar{m}^i, \mathbf{I}_S) \cap N^i(\bar{p})$. Then, for some $\tilde{\theta}^{-i}, (p, \tilde{\theta}^{-i}) \in \mathcal{S}^i(\theta; \bar{m}^i, \mathbf{I}_S)$. By construction, $P = R^{\mathsf{T}} p \in N^i(\bar{P})$ and $(R^{\mathsf{T}} p, (R_S^{-1}\tilde{\theta}^j, 0)_{j \neq i}) \in \mathcal{S}^i((R_S^{-1}\theta, 0); \bar{M}^{-i}, R)$. By Lemma 4,

$$\mathcal{P}^i\left(\begin{pmatrix} (R_S)^{-1}\theta \\ 0 \end{pmatrix}; \bar{M}^{-i}, R\right) = \mathcal{P}^i(\Theta; \bar{M}^{-i}, R), \quad (96)$$

so $R^{\mathsf{T}} p \in \mathcal{P}^i(\Theta; \bar{M}^{-i}, R)$. As before, by construction, $P \neq P^i(\Theta)$ and $P \in N^i(\bar{P}) \cap \mathcal{P}^i(\Theta; \bar{M}^{-i}, R)$, which is impossible. □

Proof of Lemma 6 For each individual, the conjecture defined by Eq. 39 is coarse and implies that

$$\pi^i(\alpha^i, \bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha})) = \boldsymbol{\pi}(\alpha^i, \bar{\alpha}^{\neg i}) \tag{97}$$

for every action α^i.

For necessity, we need to show that $\bar{\alpha}^i \in B^i(\bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha}), \delta^i)$. If this is not the case, then one can find α^i such that

$$v^i(\alpha^i, \pi^i(\alpha^i, \bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha})) > v^i(\bar{\alpha}^i, \pi^i(\bar{\alpha}^i, \bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha})). \tag{98}$$

By Eq. 97, this means that $v^i(\alpha^i, \boldsymbol{\pi}(\alpha^i, \bar{\alpha}^{\neg i})) > v^i(\bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha}^i, \bar{\alpha}^{\neg i}))$, and, hence, that $\bar{\alpha}^i$ is not stable for i, given $\bar{\alpha}^{\neg i}$ and the conjecture $(\alpha^i, \bar{\alpha}) \mapsto \boldsymbol{\pi}(\alpha^i, \bar{\alpha}^{\neg i})$, which is impossible.

For sufficiency, we need to show that each $\bar{\alpha}^i$ is stable for i, given $\bar{\alpha}^{\neg i}$ and the conjecture $(\alpha^i, \bar{\alpha}) \mapsto \boldsymbol{\pi}(\alpha^i, \bar{\alpha}^{\neg i})$. Again, if this is not true, we can fix α^i such that $v^i(\alpha^i, \boldsymbol{\pi}(\alpha^i, \bar{\alpha}^{\neg i})) > v^i(\bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha}^i, \bar{\alpha}^{\neg i}))$, and, thus, that

$$v^i(\alpha^i, \pi^i(\alpha^i, \bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha})) > v^i(\bar{\alpha}^i, \pi^i(\bar{\alpha}^i, \bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha})), \tag{99}$$

by Eq. 97. As before, this is a contradiction, since $\bar{\alpha}^i \in B^i(\bar{\alpha}^i, \boldsymbol{\pi}(\bar{\alpha}), \delta^i)$. □

Proof of Lemma 7 If $\bar{q}^i_s > 0$ and $b^i_s = 0$, the result is immediate. Now, in any other case,

$$\bar{\pi}^i_s((b^i, q^i), (\bar{b}^i, \bar{q}^i), \bar{p}) = \varphi^i_s(b_s, \bar{p}_s, e^i_s + \frac{\bar{b}^i_s}{\bar{p}_s} - \bar{q}^i_s) - m^i_s q^i_s, \tag{100}$$

and the result follows from Eq. 49. □

Proof of Lemma 8 For the first part, suppose that, contrary to the statement, there exists an action $(\hat{b}^i, \hat{q}^i)$ such that

$$v^i((\hat{b}^i, \hat{q}^i), \hat{s}) > v^i((\hat{b}^i, \hat{q}^i), p) \tag{101}$$

where $\hat{p} = \bar{\pi}^i((\hat{b}^i, \hat{q}^i), (b^i, q^i), p)$.[41] Let consumption $\hat{c}^i_s := e^i_s + \hat{b}^i_s / \hat{p}_s - \hat{q}^i_s$, for each state s, and notice that, from Lemma 7, $\hat{p} = p + \bar{m}^i(\hat{c}^i - c^i)$, so

$$\begin{aligned} -(p + \bar{m}^i(\hat{c}^i - c^i)) \cdot (\hat{c}^i - e^i) + u^i(\hat{c}^i) &= v^i((\hat{b}^i, \hat{q}^i), \hat{s}) \\ &> v^i((\hat{b}^i, \hat{q}^i), p) \\ &= -p \cdot (c^i - e^i) + u^i(c^i), \end{aligned}$$

which is impossible.

[41] Here, we are using the fact that $\bar{\pi}^i((b^i, q^i), (b^i, q^i), p) = p$.

For the second part, suppose that there exists a trade $\hat{\theta}^i$ such that

$$-\hat{p} \cdot \hat{\theta}^i + u^i(e^i + \hat{\theta}^i) > -p \cdot \theta^i + u^i(e^i + \theta^i), \tag{102}$$

where $\hat{p} = p + \bar{m}^i(\hat{\theta}^i - \theta^i)$. If this is the case, define action $(\hat{p}_s(\hat{t}heta^i_s)_+, (\hat{\theta}^i_s)_-)_{s=1}^S$ and notice that, again from Lemma 7, $\hat{p} = \bar{\pi}^i_s((\hat{b}^i, \hat{q}^i), (b^i, q^i), p)$. Then,

$$\begin{aligned} v^i((\hat{b}^i, \hat{q}^i), \bar{\pi}^i_s((\hat{b}^i, \hat{q}^i), (b^i, q^i), p)) &= -\hat{p} \cdot \hat{\theta}^i + u^i(e^i + \hat{\theta}^i) \\ &> -p \cdot \theta^i + u^i(e^i + \theta^i) \\ &= v^i((\hat{b}^i, \hat{q}^i), p), \end{aligned}$$

which is impossible. □

Proof of Lemma 9 For the first part, fix, for each $j \neq i$, θ^j stable for j given $(p, \bar{m}^j)$, such that $\theta^i + \sum_{j \neq i} \theta^j = 0$. Define (b, q) be the profile of actions associated to $(\theta^1, \ldots, \theta^I)$ at prices p. By the first part of Lemma 8, each (b^j, q^j) is stable, for $j \neq i$, given $(p, \bar{\pi}^j)$. By construction, $\boldsymbol{\pi}(b, q) = p$, so $p \in \Pi^i((b^i, q^i), \bar{\pi}^{\neg i})$.

The proof of the second part is similar, using part 2 of Lemma 8. □

Appendix A2: Existence and determinacy of equilibrium in elementary securities

A2.1 Existence

We now specialize the proof of Theorem 1 in Weretka (2007a), for the particular case treated here. All the arguments in this appendix are given for an economy with elementary securities, $\{u, e, \mathbf{I}_S\}$ satisfying our assumptions about preferences and endowments. Since we will only consider positive definite and diagonal price impact matrices, we identify these matrices with vectors in $\mathbf{R}^S_{++}$.

Let $h : (\mathbf{R}^S_+, \mathbf{R}^S_{++})^{I-1} \to \mathbf{R}^S_{++}$ be the (component-wise) harmonic mean, divided by $(I-1)$; that is,

$$h_s(m, v) = \left(\sum_j (m^j_s + v^j_s)^{-1} \right)^{-1}. \tag{103}$$

Function h is continuous, and it is well known that

$$(I-1) h_s(m, v) \leq (I-1)^{-1} \sum_j (m^j_s + v^j_s), \tag{104}$$

for all m, all v and all s.

Lemma 10 *Let γ_u be any strictly positive number, and let $(m, v) \in (\mathbf{R}^S_+, \mathbf{R}^S_{++})^{I-1}$. Suppose that for some s, $m^j_s \leq (I-2)^{-1}\gamma_u$ and $v^j_s \leq \gamma_u$ for all j. Then, $h_s(m, v) \leq (I-2)^{-1}\gamma_u$.*

Proof This is immediate, since

$$(I-1)h_s(m,v) \leq \frac{\sum_j m_s^j}{I-1} + \frac{\sum_j v_s^j}{I-1} \leq \frac{\gamma_u}{I-2} + \gamma_u. \tag{105}$$

□

Define, for each state s, $\mu_s = \max_i\{\partial u_s^i(e_s^i)\}$, a strictly positive real number. By the Inada conditions, we can fix, for each trader i and state s, a revenue transfer $\hat{\theta}_s^i$ such that $\partial u_s^i(e_s^i + \hat{\theta}_s^i) = \mu_s$. By concavity and Inada, $-e_s^i < \hat{\theta}_s^i \leq 0$.

Construct the (truncation) set of trades

$$\mathcal{T} = \{\theta \in (\mathbf{R}^S)^I : \sum_i \theta_s^i = 0 \text{ and } \theta_s^i \geq \frac{1}{2}(\hat{\theta}_s^i - e_s^i) \text{ for all } i \text{ and all } s\}, \tag{106}$$

which is nonempty, convex and compact.

Define the function $V : \mathcal{T} \to (\mathbf{R}_{++}^S)^I$, componentwise, by letting $V_s^i(\theta) = -\partial^2 u_s^i(e_s^i + \theta_s^i)$.[42] Since $\mathcal{T}$ is compact, V is continuous and I and S are finite, we can further define real numbers $\gamma_u = \max_{\theta\in\mathcal{T}} \max_{i,s} V_s^i(\theta)$ and $\gamma_d = \min_{\theta\in\mathcal{T}} \min_{i,s} V_s^i(\theta)$; these numbers, by construction, satisfy that $0 < \gamma_d \leq \gamma_u$.

Define the function $\mathcal{U} : ([0,(I-2)^{-1}\gamma_u]^S)^I \to (\mathbf{R}^S)^I$ by

$$\mathcal{U}(m) = \arg\max_{\theta\in\mathcal{T}} \sum_{i,s} \left(u_s^i(e_s^i + \theta_s^i) - \frac{1}{2} m_s^i (\theta_s^i)^2 \right), \tag{107}$$

which is well defined since $\mathcal{T}$ is compact and convex, each u^i is continuous and strongly concave, and each $m_s^i \geq 0$. Define also the function $H : ([\gamma_d,\gamma_u]^S)^I \times ([0,(I-2)^{-1}\gamma_u]^S)^I \to (\mathbf{R}^S)^I$, by letting $H^i(m,v) = h(m^{-i}, v^{-i})$.

Finally, define the function $\mathcal{F} : \mathcal{T} \times ([\gamma_d,\gamma_u]^S)^I, ([0,(I-2)^{-1}\gamma_u]^S)^I \to (\mathbf{R}^S)^I \times (\mathbf{R}^S)^I \times (\mathbf{R}^S)^I$, by

$$\mathcal{F}(\theta, v, m) = \begin{pmatrix} \mathcal{U}(m) \\ V(\theta) \\ H(m,v) \end{pmatrix}. \tag{108}$$

Lemma 11 *There exists* $(\bar{\theta}, \bar{v}, \bar{m}) \in \mathcal{T} \times ([\gamma_d,\gamma_u]^S)^I \times ([0,(I-2)^{-1}\gamma_u]^S)^I$ *such that* $\mathcal{F}(\bar{\theta}, \bar{v}, \bar{m}) = (\bar{\theta}, \bar{v}, \bar{m})$.

Proof By construction, $\mathcal{U}$ maps into $\mathcal{T}$, and V into $([\gamma_d,\gamma_u]^S)^I$. By Lemma 10, H maps into $([0,(I-2)^{-1}\gamma_u]^S)^I$. It follows that $\mathcal{F}$ maps a convex and compact set continuously into itself, so it has a fixed point. □

Let a fixed point, $(\bar{\theta}, \bar{v}, \bar{m})$, of $\mathcal{F}$ be fixed.

[42] This function is well defined, since, by construction, $(\hat{\theta}_s^i - e_s^i)/2 > -e_s^i$, so $e_s^i + \theta_s^i > 0$.

Lemma 12 *For all i and all s, it is true that $\bar{\theta}_s^i > \frac{1}{2}(\hat{\theta}_s^i - e_s^i)$.*

Proof Suppose not: suppose that for some i and some s, $\bar{\theta}_s^i = \frac{1}{2}(\hat{\theta}_s^i - e_s^i)$. Since $\hat{\theta}_s^i \leq 0$, it follows that $\bar{\theta}_s^i < 0$, which implies that, for some $j \neq i$, $\bar{\theta}_s^j > 0$. By the definition of function $\mathcal{U}$, it follows, then, that

$$\partial u_s^i(e_s^i + \bar{\theta}_s^i) - \bar{m}_s^i \bar{\theta}_s^i \leq \partial u_s^j(e_s^j + \bar{\theta}_s^j) - \bar{m}_s^j \bar{\theta}_s^j, \tag{109}$$

which is impossible, since

$$\partial u_s^i(e_s^i + \bar{\theta}_s^i) - \bar{m}_s^i \bar{\theta}_s^i = \partial u_s^i\left(\frac{1}{2}(e_s^i + \hat{\theta}_s^i)\right) - \bar{m}_s^i \bar{\theta}_s^i > \mu_s, \tag{110}$$

whereas

$$\partial u_s^j(e_s^j + \bar{\theta}_s^j) - \bar{m}_s^j \bar{\theta}_s^j < \partial u_s^j(e_s^j) \leq \mu_s. \tag{111}$$

□

We can now conclude that given preferences and endowments $\{u, e\}$, there exists an equilibrium in elementary securities $(\bar{p}, \bar{\theta}, \bar{m})$ satisfying that all price impact matrices are diagonal.

To see this, notice that, given concavity of all preferences and since $\bar{m}_s^i \geq 0$ for all i and all s, it follows from Lemma 12 that $\bar{\theta}$ actually solves program

$$\max_{\theta : \sum_i \theta^i = 0} \sum_{i,s} \left(u_s^i(e_s^i + \theta_s^i) - \frac{1}{2}\bar{m}_s^i (\theta_s^i)^2 \right). \tag{112}$$

The first-order conditions of this problem immediately imply that there exist Lagrange multipliers $\bar{p} \in \mathbf{R}^S$ such that, for all i and all s,

$$\partial u_s^i(e_s^i + \bar{\theta}_s^i) - \bar{m}_s^i \bar{\theta}_s^i = \bar{p}_s. \tag{113}$$

Identifying $\bar{m}^i$ and $\bar{v}^i$ with the diagonal, positive definite matrices

$$\begin{pmatrix} \bar{m}_1^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{m}_S^i \end{pmatrix} \text{ and } \begin{pmatrix} \bar{v}_1^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{v}_S^i \end{pmatrix}, \tag{114}$$

respectively, we get that: (i) $\sum_i \bar{\theta}^i = 0$; (ii) for each trader i, $\partial u^i(e^i + \bar{\theta}^i) - \bar{m}^i \bar{\theta}^i = \bar{p}$; (iii) for each trader i, $\bar{m}^i = (I-1)^{-1}\mathcal{H}(\bar{m}^{-i} + \bar{v}^{-i})$; and (iv) for each trader i, $\bar{v}^i = -\partial^2 u^i(e^i + \bar{\theta}^i)$.

Then, it is immediate from Lemma 1 that $(\bar{p}, \bar{\theta}, \bar{m})$ is an equilibrium.

A2.2 Generic determinacy

Assume now that, moreover, each Bernoulli function u_s^i is of class $\mathbf{C}^3$ and satisfies that $\partial^2 u_s^i$ is bounded strictly below zero.

Denote by $\mathbf{U}$ the class of Bernoulli functions that satisfy all the assumptions we have imposed,[43] and endow this set with the topology of $\mathbf{C}^3$ uniform convergence on compacta.[44] The space of economies we now consider is the set $(\mathbf{R}_{++}^S)^I \times (\mathbf{U}^S)^I$, endowed with the product topology.

We want to show that, in a dense subset of these economies, all equilibria in elementary securities are locally unique. To do this, fix an economy $\{\bar{u}, \bar{e}\}$ and an open neighborhood $\mathcal{O}$ of that economy. We need to show that for at least one economy $\{u, e\}$ in $\mathcal{O}$ the property of local uniqueness holds true.

A2.2.1 A finite-dimensional subspace of economies

As the space of economies is an infinite-dimensional manifold, it is convenient to do our analysis in a finite-dimensional, local subspace of economies. The strategy is to show that in this subspace we can find the economy $\{u, e\}$, where local uniqueness holds, arbitrarily close to $\{\bar{u}, \bar{e}\}$. In this section, we define the subspace of economies.

Fix $\epsilon > 0$, and a $\mathbf{C}^\infty$ function $\rho : \mathbf{R}_{++} \to \mathbf{R}_+$ such that

$$\rho(x) = \begin{cases} 1, \text{ if } x < \max_s\{\sum_i \bar{e}_s^i\} + I\epsilon + 1; \\ 0, \text{ if } x \geq \max_s\{\sum_i \bar{e}_s^i\} + I\epsilon + 2. \end{cases} \tag{115}$$

Also, define for each i and s the "perturbed Bernoulli" function $u_s^i : \mathbf{R}_{++} \times \mathbf{R} \to \mathbf{R}$ by

$$u_s^i(x_s^i, \delta_s^i) = \bar{u}_s^i(x_s^i) + \frac{1}{2}\rho(x_s^i)\delta_s^i(x_s^i)^2. \tag{116}$$

Since each $\max_s\{\sum_i \bar{e}_s^i\}$ is finite, there exists $\bar{\delta} > 0$ such that for all $-\bar{\delta} < \delta_s^i < \bar{\delta}$, function $u_s^i(\cdot, \delta_s^i) \in \mathbf{U}$.

Now, we consider the finite dimensional subset of economies defined by endowments $e \in B_\epsilon(\bar{e})$, and preferences given, for each i and s, by $u_s^i(\cdot, \delta_s^i)$ for some $\delta_s^i \in B_{\bar{\delta}}(0)$. This is a submanifold parameterized by $B_\epsilon(\bar{e}) \times B_{\bar{\delta}}(0)^{SI}$. For notational simplicity, we identify the vector $\delta^i = (\delta_s^i)_s$ with the diagonal matrix $\text{diag}(\delta_s^i)$, and denote $\delta = (\delta^i)_{i=1}^I$. Similarly, we again identify the vector $m^i = (m_s^i)_s$ with the diagonal matrix $\text{diag}(m_s^i)$.

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[43] Namely, that each u_s^i is differentiably strictly monotonic and differentiably strictly concave, and satisfies Inada conditions. Unlike in Proposition 5, we do not require now that these functions be CRRA.

[44] See, for instance, Aliprantis and Border (1999).

A2.2.2 Transversality

Now, consider the function $\mathcal{G} : \mathbf{R}^S \times (\mathbf{R}^S)^I \times (\mathbf{R}^S_{++})^I \times B_\epsilon(\bar{e}) \times B_{\bar{\delta}}(0) \to (\mathbf{R}^S)^I \times \mathbf{R}^S \times (\mathbf{R}^S)^I$, defined by

$$\mathcal{G}(p, \theta, m, e, \delta) = \begin{pmatrix} \vdots \\ p + m^i \theta^i - \partial_x u^i(e^i + \theta^i, \delta^i) \\ \vdots \\ \sum_i \theta^i \\ \vdots \\ (m^i_s \sum_{j \neq i} (m^j_s - \partial^2_{x,x} u^j_s(e^j_s + \theta^j_s, \delta^j_s)^{-1}) - 1)_s \\ \vdots \end{pmatrix}. \tag{117}$$

It follows from Lemma 1 that (p, θ, m) is an equilibrium in elementary securities for economy (e, δ) if, and only if, $\mathcal{G}(p, \theta, m, e, \delta) = 0$.

Lemma 13 *Function $\mathcal{G}$ is transverse to 0. That is, whenever $\mathcal{G}(p, \theta, m, e, \delta) = 0$, matrix $\partial\mathcal{G}(p, \theta, m, e, \delta)$ has full (row) rank.*

Proof Let $(p, \theta, m, e, \delta)$ be such that $\mathcal{G}(p, \theta, m, e, \delta) = 0$. Since each $\theta^i_s > -e^i_s$ and $\sum_i \theta^i = 0$, it follows that each $\theta^i_s \leq \sum_j e^i_s < \sum_j \bar{e}^j_s + I\epsilon$, and then, by construction, it follows that for any x in a neighborhood of $e^i + \theta^i$,

$$u^i(x, \delta^i) = \bar{u}^i(x) + \frac{1}{2} x^\top \delta^i x. \tag{118}$$

By direct computation, and considering only the derivatives with respect to $(\theta^1, \ldots, \theta^I, e^1)$ and $(m^1, \ldots, m^I, \delta^1, \ldots, \delta^I)$, matrix $\partial\mathcal{G}(p, \theta, m, e, \delta)$ can be written in partition form as

$$\Gamma = \begin{pmatrix} \Gamma_{1,1} & \Gamma_{1,2} \\ \Gamma_{2,1} & \Gamma_{2,2} \end{pmatrix}, \tag{119}$$

where[45] $\Gamma_{2,2} = (\Gamma^L_{2,2}\ \Gamma^R_{2,2})$,

$$\Gamma_{1,1} = \begin{pmatrix} m^1 - \partial^2 \bar{u}^1 - \delta^1 & \cdots & 0 & -\partial^2 \bar{u}^1 - \delta^1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & m^I - \partial^2 \bar{u}^I - \delta^I & 0 \\ \mathbf{I} & \cdots & \mathbf{I} & 0 \end{pmatrix}, \tag{120}$$

[45] The precise form of matrix $\Gamma_{2,1}$ is immaterial for computations below.

$$\Gamma_{1,2} = \begin{pmatrix} \mathrm{diag}(\theta_s^1) & \cdots & 0 & -\mathrm{diag}(\theta_s^1) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{diag}(\theta_s^I) & 0 & \cdots & -\mathrm{diag}(\theta_s^I) \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{121}$$

$$\Gamma_{2,2}^L = \begin{pmatrix} \sum_{j \neq 1}(m^j - \partial^2 \bar{u}^j - \delta^j)^{-1} & \cdots & -\mathrm{diag}(\frac{m_s^1}{(m_s^I - \partial^2 \bar{u}_s^I - \delta_s^I)^2}) \\ \vdots & \ddots & \vdots \\ -\mathrm{diag}(\frac{m_s^I}{(m_s^1 - \partial^2 \bar{u}_s^1 - \delta_s^1)^2}) & \cdots & \sum_{j \neq I}(m^j - \partial^2 \bar{u}^j - \delta^j)^{-1} \end{pmatrix} \tag{122}$$

and

$$\Gamma_{2,2}^R = \begin{pmatrix} 0 & \cdots & \mathrm{diag}(\frac{m_s^1}{(m_s^I - \partial^2 \bar{u}_s^I - \delta_s^I)^2}) \\ \vdots & \ddots & \vdots \\ \mathrm{diag}(\frac{m_s^I}{(m_s^1 - \partial^2 \bar{u}_s^1 - \delta_s^1)^2}) & \cdots & 0 \end{pmatrix}. \tag{123}$$

Notice that matrix $\Gamma_{1,1}$ is nonsingular, by standard arguments, given our assumptions on concavity of preferences and the fact that each price impact matrix is positive definite.[46] Then, to see that Γ has full row rank, it suffices to observe that there exists an $(SI + S + SI + SI) \times SI$ matrix A such that

$$\Gamma A = \begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix}. \tag{125}$$

For this, just let matrix

$$A = \begin{pmatrix} 0 \\ A_2 \end{pmatrix}, \tag{126}$$

[46] To see this, the argument is the one that gives determinacy in competitive models: (i) notice that the submatrix composed by the first I superrows and I supercolumns is nonsingular, since it is a diagonal matrix of positive numbers: each $m_s^i > 0$ and each $\partial^2 \bar{u}_s^i(e_s^i + \theta_s^i) + \delta_s^i < 0$, as $u_s^i(\cdot, \delta_s^i)$ is strongly concave; (ii) Now, to see that the whole $\Gamma_{1,1}$ is nonsingular, it suffices to observe that if one constructs the $S(I+1) \times S$ matrix A by letting the first $S \times S$ superrow be $\mathbf{I}$, letting the last $S \times S$ superrow be $-(\partial^2 \bar{u}^1(e^i + \theta^i) + \delta^1)^{-1}(m^1 - \partial^2 \bar{u}^1 - \delta^1)$, and letting every other element be 0, we get, by direct computation, that

$$\Gamma_{1,1} A = \begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix}. \tag{124}$$

where

$$A_2 = \begin{pmatrix} m^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & m^I \\ m^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & m^I \end{pmatrix}. \tag{127}$$

By direct computation, the result follows from the fact that, for each i,

$$m^i \sum_{j \neq i} (m^j - \partial^2 \bar{u}^j (e^j + \theta^j) - \delta^j)^{-1} = \mathbf{I} \tag{128}$$

□

Now, we argue that for almost all economies in the finite-dimensional subspace of economies all equilibria are locally isolated.

Lemma 14 *For a full-measure subset of $B_\epsilon(\bar{e}) \times B_{\bar{\delta}}(0)^{SI}$, it is true that for any equilibrium in elementary securities $(\bar{p}, \bar{\theta}, \bar{m})$ there exists an open neighborhood in which there are no more equilibria for the same economy.*

Proof By Lemma 13 and the Transversality Theorem (see Guillemin and Pollack 1974), it follows that for any (e, δ) in a full measure subset of $B_\epsilon(\bar{e}) \times B_{\bar{\delta}}(0)^{SI}$, function $\mathcal{G}(\cdot, e, \delta)$ is transverse to 0. If $(\bar{p}, \bar{\theta}, \bar{m})$ is an equilibrium for (e, δ), it must be that $\mathcal{G}(\bar{p}, \bar{\theta}, \bar{m}, e, \delta) = 0$, and hence that $\partial_{p,\theta,m}\mathcal{G}(\bar{p}, \bar{\theta}, \bar{m}, e, \delta)$ has full row rank. But $\partial_{p,\theta,m}\mathcal{G}$ is a square matrix, so $\partial_{p,\theta,m}\mathcal{G}(\bar{p}, \bar{\theta}, \bar{m}, e, \delta)$ is nonsingular and hence, by the inverse function theorem, $\mathcal{G}(\cdot, e, \delta)$ is a bijection in a neighborhood of $(\bar{p}, \bar{\theta}, \bar{m})$. □

A2.2.3. Denseness

The last lemma shows that in almost every economy in the finite-dimensional subspace, the property of local uniqueness holds. To complete our argument that the set of economies where the property holds is dense in the general space of economies, it suffices to show that the finite subspace of economies intersects the pre-fixed neighborhood $\mathcal{O}$ of $\{\bar{u}, \bar{e}\}$. But this is true, because the topology with which the space of economies is endowed is metrizable (see Villanacci et al. 2002, p. 428), and in this metric the distance from $\bar{u}_s^i = u_s^i(\cdot, 0)$ to $u_s^i(\cdot, \delta_s^i)$ can be made arbitrarily small by choosing a small enough $\delta_s^i > 0$.

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